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COLLECTIVE EXCITATIONS OF NUCLEAR MATTER

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June 23, 1958

Abstract

A study of the collective motions of nuclear matter has been made. We first give a purely classical macroscopic description of hydrodynamic waves in nuclear matter, and suggest some experimental consequences of their excitation. Next a quantum mechanical study of the collective eigenstates of nuclear matter is taken up. The starting point of this discussion is the theory of the nuclear ground state as given by Brueckner and his collaborators. The excited states are described by means of the method developed by Sawada to apply to an electron gas. We generalize this method so as to include the internal degrees of freedom associated with spin and i-spin and to handle the momentum dependence of the level-shift operator K used by Brueckner. The connection between the quantum-mechanical eigenstates and the classical hydrodynamic motion is established. As a consequence of the internal degrees of freedom, there exist not only the usual compressive waves, but spin, i-spin, and coupled spin-i-spin waves. The i-spin waves can be associated with the Goldhaber-Teller oscillations.

We have investigated the corrections to the Sawada theory. This gives rise to the damping of the stable Sawada collective eigenmodes, analogous to the viscous damping of a plasma oscillation. In some cases, however, we find not damped but exponentially growing waves. This seems to correspond to the system's collapsing on itself. This difficulty must lie in our description of the ground state and we can at the moment only speculate on the origin of this difficulty.

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1. Introduction

The shell model has provided the general framework in terms of which most nuclear models are now described. It is characteristic of this model that orbits of single nucleons are rather simply related to states of nuclei. This includes excited nuclear states which are considered to arise from single-particle excitations. We need not review here the considerable successes of this point of view--or its limitations. In the latter connection, however, we recall that the failure of the shell model to correctly predict nuclear magnetic moments, quadrupole moments, and some excited states of heavy nuclei has been generally interpreted as due to collective motions of nuclear particles. To improve the shell-model description, Bohr and Mottelson, Hill and Wheeler, and others have introduced collective oscillations of nuclear matter.[1] Also Goldhaber and Teller[2] have proposed a relative motion of the neutron and proton "fluid components" in nuclei as an explanation of the "giant resonance" in the interaction of nuclei with γ rays of about 20 Mev energy.

The notion of cooperative effects in nuclei is hardly new to nuclear physics. The liquid-drop model of Bohr[3] exploited a hydrodynamic analogy. His argument followed closely that of the kinetic theory of gas hydrodynamics. That is, if a given nucleon is strongly scattered by its neighbors, any local excitation will be shared by many nucleons, and cooperative "hydrodynamic" motion is expected. On the other hand, if the nucleon is not strongly scattered, cooperative motion does not occur and there is no hydrodynamic motion. [In kinetic theory this is the distinction between the Boyle and the Knudson gas.]

Because of the success of the shell model it has often been argued that the mean free path for collisions between nucleons in nuclei is too large to justify the assumption that hydrodynamic motion can occur. We feel, however, that this argument may be fallacious. First, the concept of a collision between two particles is not precisely defined for a medium in which several particles may interact simultaneously. [That this point is not trivial is evidenced by the fact that it involves questions of principle that are still not understood in kinetic transport theory.[4]] Indeed, to take an extreme case, hydrodynamic motion may obtain in the complete absence of "collisions" between particles, if instead the particles interact with a "collective field" produced by the motion. This is illustrated by the well-known phenomenon of plasma oscillations in an electron gas. These represent a definite hydrodynamic mode of motion by which the electrons interact directly with the electric field caused by the cooperative motion and not with one another "individually."[5]

In this paper we pursue the argument just given to investigate possible modes of hydrodynamic motion in "nuclear matter"--or a nuclear medium of infinite extent. Undoubtedly boundary conditions at the surface of actual nuclei will modify the details of our conclusions; on the other hand, it is hoped that some physical insight into the mechanism of cooperative motions may be obtained from these considerations.

The analogy to plasma oscillations, which originally motivated this study, turns out to be very helpful.[6] The quantitative results differ considerably, however, from those for an electron gas. This is associated in large part with the fact that nuclear forces have a finite range, whereas plasma oscillations are due to long-range Coulomb interactions.[7]

In Section 2 we give an entirely classical, macroscopic derivation of nuclear hydrodynamic motion, including possible mention of experimental observation. This derivation depends on the assumption that the nuclear volume energy is a minimum at observed nuclear densities. It ignores, however, the important question of the damping of the motion obtained.

Following this classical study of nuclear hydrodynamic oscillations, a detailed quantum-mechnical treatment of these phenomena is presented. The starting point of the discussion is the theory of the nuclear ground-state structure as formulated by Brueckner and his collaborators.[8] The methods developed by Gell-Mann and Brueckner,[9] Sawada et al.[10], and Wentzel[11] are applied to describe the spectrum of excited states, which includes the hydrodynamic eigenmodes.

To be more specific, we first generalize the method to systems of particles having internal degrees of freedom and to "scatterings" described by the level-shift operators K, employed by Brueckner et al.[8] We then study in detail the macroscopic hydrodynamic motion (as a function of time) that arises from "wave packets" of the hydrodynamic eigenmodes. In doing this, we shall see the close relation to the purely classical hydrodynamic discussion of Section 2. Because of the spin and isotopic-spin degrees of freedom of nucleons, we find four classes of hydrodynamic motion. The simplest is a purely hydrodynamic mode (sound waves) involving density variations. In addition, there are spin-wave solutions, corresponding to periodic oscillations of the local spin density, Goldhaber-Teller oscillations, and coupled spin and i-spin waves.

The above discussion indicates that the Sawada method can be applied to a variety of problems involving cooperative fluid motions. Since this technique represents only an approximate solution to the many-particle problem, it is necessary to discuss also corrections to the Sawada method. To do this, we have employed a time-dependent Schrödinger equation and considered the excitation of hydrodynamic motion as a transient problem. The corrections to the Sawada treatment then appear in the form of damping of the simple hydrodynamic motion--or as a mechanism leading to ergodic behavior of the many-particle system. By using a time-dependent approach, we avoid some exceedingly delicate problems concerning true eigenstates of multiparticle systems.

With the above technique we are able to discuss the damping of the collective motion. This appears in a manner analogous to the viscous damping of plasma oscillations. In some cases one finds not damped but exponentially growing waves. This instability seems to have a simple origin, occurring for systems that are at too low a density to satisfy saturation conditions. The exponential growth then seems to correspond to a collapse of the system into droplets of higher density.

The simple compressive mode, described above, appears to be unsuitable in this sense when one uses the Brueckner ground-state density and level-shift operators.

2. Classical Development of Hydrodynamic Motion

We consider a large nucleus and imagine that we make a small displacement from equilibrium $\xi(r, t)$ of the nuclear matter at the point r. We suppose that ξ varies sufficiently slowly with r that a large number of nucleons are involved; thus, we may apply classical mechanics to the subsequent motion.

As we perform the displacement, the average nucleon velocity at r is

$$v = \frac{d\xi}{dt} = \frac{\partial \xi}{\partial t} \tag{2-1}$$

if ξ is small. Let the mass density of nuclear matter be

$$\rho(r, t) = \rho_0 + \rho'(r, t),$$

where ρ' is the (small) deviation from the equilibrium density, ρ_0. The continuity equation for ρ is

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0.$$

If we make use of the assumed smallness of v and ρ', this may be approximately rewritten as

$$\frac{\partial \rho'}{\partial t} = -\rho_0 \nabla \cdot v = -\rho_0 \nabla \cdot \frac{\partial \xi}{\partial t}$$

or

$$\rho' = -\rho_0 \nabla \cdot \xi . \tag{2-2}$$

Now, the displacement ξ will be resisted by a restoring force per unit mass F, since the nucleus was originally assumed to be in equilibrium. Thus, by Newton's equation of motion, we have

$$\rho_0 \frac{dv}{dt} = \rho_0 F , \tag{2-3}$$

which is correct to first order in small quantities. Let us now take F to be derivable from a potential Φ :

$$F = -\nabla \Phi . \tag{2-4}$$

Following the reasoning of Berg and Wilits,[12, 13] we expect $\underline{\Phi}$ to be determined by ρ', if ρ_0 is the equilibrium density. For example, we might expect the form

$$\underline{\Phi} = \frac{1}{\rho_0} \left[C_1 \rho' + C_2 \nabla^2 \rho' + C_3 \nabla^4 \rho' + \dots \right], \tag{2-5}$$

where C_1, C_2, ... are constants, since terms of $O(\rho'^2)$ are negligible by our assumption that the displacement $\underset{\sim}{\xi}$ is small. This expression is consistent with the assumption of small displacements, which means that only terms linear in ρ' need be retained. If we restrict ourselves to disturbances that also vary slowly (presumably over the range of nuclear forces), then Eq. (2-5) may be replaced by

$$\underline{\Phi} = \frac{C_1}{\rho_0} \rho' . \tag{2-6}$$

Using Eqs. (2-2) and (2-4), we have

$$\begin{aligned} \underset{\sim}{F} &= + C_1 \underset{\sim}{\nabla} \underset{\sim}{\nabla} \cdot \underset{\sim}{\xi} , \\ \underline{\Phi} &= - C_1 \underset{\sim}{\nabla} \cdot \underset{\sim}{\xi} . \end{aligned} \tag{2-7}$$

Since the disturbance must be caused by an external force, say $\underset{\sim}{F}_0$, this must be included in the first of Eqs. (2-7), which is now rewritten as

$$\underset{\sim}{F} = C_1 \underset{\sim}{\nabla} \underset{\sim}{\nabla} \cdot \underset{\sim}{\xi} + \underset{\sim}{F}_0 . \tag{2-8}$$

This external force may be due to the passage of a fast particle through the nucleus.

With this development, the equation of motion (2-3) is now

$$\frac{\partial^2 \underset{\sim}{\xi}}{\partial t^2} = C_1 \underset{\sim}{\nabla} \underset{\sim}{\nabla} \cdot \underset{\sim}{\xi} + F_0 . \tag{2-9}$$

Let us set

$$C_1 \equiv a^2 , \tag{2-10}$$

and take the divergence of (2-9):

$$\left[\frac{\partial^2}{\partial t^2} - a^2 \nabla^2\right] \nabla \cdot \xi = \nabla \cdot F_0 . \tag{2-11}$$

We may also rewrite this as

$$\left[\frac{\partial^2}{\partial t^2} - a^2 \nabla^2\right] \Phi = - a^2 \nabla \cdot F_0 \equiv - S . \tag{2-12}$$

We have abbreviated $a^2 \nabla \cdot F_0$ as S, the "source" of the disturbance of the nuclear matter.

In the absense of the source term, Eq. (2-12) is a simple wave equation, describing acoustic waves in the nucleus. These waves travel with the "sound speed" a.

Our derivation has been oversimplified in two respects. First we have neglected the other degrees of freedom associated with the spin and isotopic spin of nucleons. This leads to the possibility of other eigenmodes of nuclear motion. We have also used the static relation (2-5) in the time-dependent equation (2-3). When the displacement takes place at a finite rate, (2-5) may be changed in form. (That is, the nucleons may make nonadiabatic transitions to excited states as the wave passes through the medium.) Equation (2-5) is not unreasonable for long-wave-length disturbances, however, because the rate at which the displacement occurs varies with the reciprocal of the wave length. Consequently, as the wave length becomes large, nonstatic corrections to (2-5) might be expected to become negligible.

In later sections we discuss the other eigenmodes of nuclear motion. These are all governed by an equation of the form (2-12), so that we may consider this to apply to any nuclear eigenmode. We shall also investigate nonadiabatic corrections to Eq. (2-5). These and many other results will follow from a general quantum-mechanical treatment of collective oscillations of nuclear matter. In the remainder of this section we simply give applications of Eq. (2-12).

We first relate the sound speed a to the nuclear compressibility. The work per unit mass done in making the displacement $\delta\xi$ is $(-)\, \delta\xi \cdot F$. The total work in a volume τ associated with the displacement ξ is then

$$W = -\rho_0 \int d\tau \int \delta \underset{\sim}{\xi} \cdot \underset{\sim}{F}$$
$$= -\rho_0 \; d\tau \underset{\sim}{\xi} \cdot \underset{\sim}{F},$$

based on the fact that the restoring force is linear in ξ. We make use of Eq. (2-8), and neglect boundary conditions, so that this expression becomes

$$W \approx \frac{\rho_0}{2} \int d\tau \, (\underset{\sim}{\nabla} \cdot \underset{\sim}{\xi})^2 a^2$$
$$\approx \frac{\rho_0}{2} \tau a^2 (\underset{\sim}{\nabla} \cdot \underset{\sim}{\xi})^2 , \tag{2-13}$$

or, according to Eq. (2-2). [M is the nucleon mass]

$$\frac{W}{\rho_0 \tau} M = \frac{M}{2\rho_0^2} a^2 (\rho')^2 . \tag{2-14}$$

This is the compressional energy per nucleon. The nuclear compressibility[12] is conventionally defined as

$$K = 9 \rho^2 \frac{\partial^2}{\partial \rho^2} \left[\frac{W}{\rho \tau} M \right] \Bigg|_{\rho = \rho_0} .$$

Using Eq. (2-14), we find the relation

$$a^2 = \frac{1}{9} \frac{K}{M} . \tag{2-15}$$

Estimated values of K range from

$$K = 187 \text{ Mev}^{9}$$

to

$$K = 302 \text{ Mev};^{13}$$

these values lead to sound velocities

$$a/c = 0.14,$$

and

$$a/c = 0.19, \tag{2-16}$$

where c is the speed of light.

A number of possible means might be employed to generate the waves described by Eq. (2-12). We mention in particular very energetic nuclear interactions, initiated by a single relativistic particle. In this case the mesons produced and knock-on nucleons are largely confined to a cone of narrow opening angle. These particles pass through the nucleus (boring a hole through it, so to speak.) The initial disturbance is confined to a line along the orbit of these particles, and the source S in Eq. (2-12) may be written as

$$S = S(\underset{\sim}{x} - \underset{\sim}{V}t), \tag{2-17}$$

where $\underset{\sim}{V}$ is the velocity of the disturbing particle or particles. For cases of interest $\underset{\sim}{V}$ is close to light velocity c. Let us suppose $\underset{\sim}{V}$ to be parallel to the z axis. By introducing the Fourier decomposition of the source,

$$S = (2\pi)^{-3} S_0 \int d^3k g(k) e^{i\underset{\sim}{k}\cdot(\underset{\sim}{x}-\underset{\sim}{V}t)}, \tag{2-18}$$

where S_0 is a constant and $g(0)=1$, we may solve the wave equation, Eq. (2-12), for Φ:

$$\Phi = -(2\pi)^{-3} S_0 \int \frac{d^3k g(k) e^{i\underset{\sim}{k}\cdot(\underset{\sim}{x}-\underset{\sim}{V}t)}}{a^2k^2 - (\underset{\sim}{k}\cdot\underset{\sim}{V})^2} . \tag{2-19}$$

In our case we are interested only in supersonic motion for which we have $V > a$. Equation (2-18) is readily integrated (subject to a retarded boundary condition and assuming long wave lengths) to give

$$\begin{aligned}\Phi &= 0 \quad \text{for } z \geq Vt \\ &= 0 \text{ for } z < Vt,\ (y^2 + x^2)\left(\frac{V^2}{a^2} - 1\right) > (z - Vt)^2 \\ &= \frac{S_0}{2\pi a^2} \frac{1}{\left[(z-Vt)^2 - \left(\frac{V^2}{a^2} - 1\right)(x^2 + y^2)\right]^{1/2}} \\ &\text{for } z < Vt,\ (y^2 + x^2)\left(\frac{V^2}{a^2} - 1\right) < (z - Vt)^2 .\end{aligned} \tag{2-20}$$

The form of this solution is rather typical of the disturbance produced by a supersonic particle. Most of the energy is localized in the neighborhood of the shock front, which is a cone of half angle β:

$$\beta = \sin^{-1}\left(\frac{a}{V}\right) . \tag{2-21}$$

The absolute value of the energy dissipated by the shock depends on a detailed knowledge of how the incident particle tunnels through the nucleus. Rather than attempt to give a description of this complicated process here, we simply characterize the strength of the interaction by the parameter S_0 which occurs in Eq. (2-18).

For this purpose, we define an energy dissipation per unit length, d, for a point P on the incident particle's path which is one nuclear radius away from the shock front. In this way we can apply Eq. (2-19), which is the solution for propagation in an infinite medium, to a finite system. We assume here that the most of the energy is dissipated when the shock front reaches the nuclear surface. The geometry of the situation is shown in Fig. 1. The energy dissipated in the shaded slab is simply d times the thickness of the slab. We obtain a crude expression for d by integrating Eq. (2-14) for the energy density W/τ over the surface of the slab. The singularity in Eq. (2-19) is smoothed over a distance of the order of the range of nuclear forces. In addition we assume that the external force per unit mass, F_0, may be represented by a potential function

$$v(\underset{\sim}{x}) = v_0 \frac{e^{-\kappa x}}{\kappa x} . \tag{2-21}$$

This potential is simply related to the source in the wave equation (2-12),

$$S = - a^2 \nabla^2 v(\underset{\sim}{x}) . \tag{2-22}$$

Because of the many approximations that have been made, the following estimate is probably reliable only to within a factor of two:

$$\kappa d \simeq (\kappa r_0)^3 \left[\frac{\bar{V}}{200 \text{ Mev}}\right]^2 \text{Mev},$$

where

$$\bar{V} = \int d^2 x \rho_0 v$$

is the volume integral of the potential, and κ^{-1} is the range of the force. One can get a fairly large energy dissipation, such as $\kappa^{-1} d \simeq 10$ Mev, by simply taking $\kappa r_0 \sim 1$ and $\bar{V} \sim 600$ Mev. A volume integral for the potential of this amount corresponds to only 50 or 100 Mev for the individual nucleon-nucleon encounters involved in the tunneling of the incident particle through the nucleus.

The propagation of a shock wave through a nucleus should lead to a number of interesting effects. When the shock front reaches the nuclear surface, nucleons or collections of nucleons (e.g. light nuclei) may be ejected and with an energy related to the sound velocity. If the sound velocity were very much larger than the internal nuclear velocities, we could expect the nucleons to be emitted just in the direction normal to the shock front.

Measured with respect to the incident direction, this characteristic emission angle is

$$\theta_0 = \frac{\pi}{2} - \beta. \qquad (2\text{-}23)$$

Such a peaked distribution will actually be smeared out for a number of reasons. The most important of these are the internal momentum distribution of the nucleons and the refractions of the ejected nucleons at the nuclear surface. Before proceeding with the calculations of the energy and angular distribution of the ejected nucleons, we should point out that the observation of this distribution would provide a measurement of the "nuclear sound velocity," which is, as we have seen, directly related to the nuclear compressibility. At present there is no accurate measurement of this quantity.

We calculate the angular and energy distribution of the nucleons emitted when the shock wave strikes the nuclear surface using the following model and assumptions.

We consider the nucleus to be a degenerate Fermi gas inside a sphere of radius R. Instead of solving the actual wave equation subject to the boundary conditions of a finite system, we simply use the solutions (2-19) for an infinite medium. Furthermore we consider only the shock front to be important in causing the emission of particles from the nuclear surface. We assume that the nucleons in the shock front possess an additional momentum $M\underset{\sim}{a}$, where $\underset{\sim}{a}$ is the velocity of the shock wave and is normal to the shock front.

When the shock front reaches the nuclear surface, we suppose that the nucleons in the shock front, which now possess this additional momentum $M\underset{\sim}{a}$, will continue on through the nuclear surface and thus be emitted. The number of particles ejected into the element d^3p ($\underset{\sim}{p}$ is the momentum of the ejected particles) is given by the following integral over the nuclear surface,

$$I(P)d^3p = d^3p\int_0^{2\pi} d\nu \int_{-1}^{+1} d(\cos\mu)\,\epsilon(\underset{\sim}{p}\cdot\underset{\sim}{\Omega})N(\underset{\sim}{P})\,\frac{\underset{\sim}{p}\cdot\underset{\sim}{\Omega}}{\sqrt{(\underset{\sim}{p}\cdot\underset{\sim}{\Omega})^2 + p_0^{\,2}}} \qquad (2\text{-}24)$$

In the integral, μ, ν are the spherical coordinates of the normal to the nuclear surface Ω, and the polar axis is the direction of the incident particle that generated the shock wave. We have restricted ourselves to the case in which the incident particle strikes the nucleus with zero impact parameter. The step function

$$\epsilon(p\cdot\Omega) = \begin{cases} 1, & p\cdot\Omega > 0 \\ 0 & p\cdot\Omega < 0 \end{cases} \tag{2-25}$$

insures that the escaping particle moves away from the nuclear surface. Here N(P) is the distribution function for nuclear momentum. The momentum of a nucleon located in the shock front is, according to our previous assumption,

$$p' = P + Ma .$$

We relate this internal momentum to the external momentum p, by reducing the normal component of p' by a fixed amount p_0 while leaving the tangential component unchanged:

$$p\cdot\Omega = p'\cdot\Omega - p_0 ,$$
$$p - (p\cdot\Omega)\Omega = p' - (p'\cdot\Omega)\Omega . \tag{2-27}$$

The momentum p_0 is of course simply related to the potential-well depth of the nucleus. The final factor in Eq. (2-24) is simply the Jacobian of the transformation in going from d^3p' to d^3p.

Equation (2-24) involves a fairly straightforward numerical integration, the details of which we will not go into here. Typical results are given in Fig. 2, where the following values have been assumed for the pertinent parameters: Fermi energy, $E_F = 40$ Mev; "normal" energy loss, $V \equiv p_0^2/2M = 50$ Mev; sound velocity, $a = 1/3$ c. Equation (2-24) can be evaluated in closed form if the effects of refraction are ignored. This result should be valid for particles of sufficiently high energy. If $I\sqrt{E}\,d\Omega dE$ is the number ejected into the solid angle $d\Omega$ about θ and the energy interval dE, we have

$$I = \cos^{-1}A + \cos^{-1}\left(\frac{|\cos\theta|}{\sqrt{1+\tan^2\theta A^2}}\right) , \tag{2-28}$$

where

$$A = f(p)\,(\sin\theta\sin\theta_0)^{-1} - \cot\theta\cot\theta_0 , \tag{2-29}$$

and

$$f(p) = \frac{E + \frac{1}{2} Ma^2 + V - E_F}{2\sqrt{\frac{1}{2} Ma^2 (E + V)}} \leq 1 . \qquad (2\text{-}30)$$

This result is valid for $f(p) > \cos\theta_0$, which is the case of interest here. Equation (2-24), and this approximation as well, can easily be generalized to include an effective mass. The results in Fig. 2 are not to be taken too literally. They do show, though, that the high-energy nucleons emitted by this mechanism will have an angular distribution peaked at an angle considerably away from the forward direction, given approximately by θ_0. This behavior differs markedly from the usual description of a high-energy nucleon-nucleus interaction, which is usually supposed to be initiated by a direct interaction and then followed by a cascade and the evaporation of particles. This latter picture predicts some very energetic particles emitted in the forward direction while the rest (and most) of the emitted nucleons are low in energy and distributed almost isotropically. This marked difference in the angular distributions resulting from these two mechanisms should make possible the identification of high-energy nucleons resulting from the excitation of nuclear shock waves by very energetic incident particles. Finally we note that the curves in Fig. 2 are undetermined by an over-all factor, which arises from our ignorance about the details of how the shock is initiated. In other words, we do not know the absolute magnitude of this effect, i.e., the energy loss per unit path length of the incident particle. The experimental observation of these effects would, of course, serve to remove this uncertainty.

3. Formulation of Quantum Mechanical Theory

We now give a quantum mechanical discussion of the collective motion based on the technique developed by Sawada et al.[10] for the electron gas. The spirit of our calculation is that of the Brueckner theory of nuclear structure.

We suppose the nuclear matter to be confined to a "large" box of volume Ω. In the absence of interactions the nucleons form a degenerate Fermi gas, their individual states being labeled by plane-wave momenta p and a spin and isotopic spin index λ. All particles have momenta less than the Fermi momentum p_F.

Next, let us imagine the nucleons to interact via two-body forces. As the interactions are "turned on," nucleons are scattered into and out of the Fermi sea. These interactions are described by the nuclear Hamiltonian

$$H = |K + |K_0 + V_s + V' \quad . \tag{3-1}$$

The term $|K_0$ represents the Brueckner ground-state energy,

$$|K_0 = \sum_{\lambda=1}^{4} \sum_{p<p_F} E_{p,\lambda} - \frac{1}{2} \sum_{\lambda,\lambda'=1}^{4} \sum_{p,p'<p_F} (\underset{\sim}{p}, \lambda; \underset{\sim}{p}', \lambda' |K| \underset{\sim}{p}, \lambda; \underset{\sim}{p}'; \lambda'), \tag{3-2}$$

with

$$E_{p,\lambda} = \frac{p^2}{2M} + \sum_{\lambda'=1}^{4} \sum_{p'<p_F} (\underset{\sim}{p}, \lambda; \underset{\sim}{p}', \lambda' |K| \underset{\sim}{p}, \lambda; \underset{\sim}{p}', \lambda') \; . \tag{3-3}$$

Here λ is an index describing the spin and isotopic-spin label of a nucleon (λ=1, 2, 3, 4) and K is the "energy shift" matrix used by Brueckner and his collaborators.[8,9] (We discuss the evaluation of these at the end of this section.) Thus $E_{p,\lambda}$ is the "effective energy" of a nucleon in nuclear matter in Brueckner's terminology. We shall assume that $E_{p,\lambda}$ is independent of λ, writing

$$E_{p,\lambda} \equiv E_p \; . \tag{3-4}$$

The "kinetic energy" in Eq. (3-1) refers to particles and holes defined with respect to a degenerate Fermi gas:

$$\mathbb{K} = \sum_{\lambda=1}^{4} \mathbb{K}_\lambda ,$$

$$\mathbb{K}_\lambda = \sum_{p > p_F} E_p\, a^*_{p,\lambda} a_{p,\lambda} - \sum_{p < p_F} E_p\, b^*_{p,\lambda}\, b_{p,\lambda} . \quad (3\text{-}5)$$

Here $a_{p,\lambda}$ and $a^*_{p,\lambda}$ respectively annihilate and create a nucleon with momentum p and "spin" λ; these quantities are defined only for $p > p_F$. Similarly, $b_{p,\lambda}$ and $b^*_{p,\lambda}$ are annihilation and creation operators for "holes" within the Fermi sea ($b_{p,\lambda} = a^*_{p,\lambda}$; $b^*_{p,\lambda} = a_{p,\lambda}$ for $p < p_F$), being defined only for $p < p_F$. Following Sawada the interaction energy V_s is taken as

$$V_s = \frac{1}{2\Omega} \sum_{\substack{q,\lambda,\lambda' \\ \lambda'_0, \lambda_0}} \sum_{p,p'} (p+q, \lambda; p'-q, \lambda' \,|\mathbb{K}|\, p, \lambda_0; p', \lambda'_0)$$

$$\times \left[a^*_{p+q,\lambda}\, b^*_{p,\lambda_0} + b_{p+q,\lambda}\, a_{p,\lambda_0} \right]$$

$$\times \left[a^*_{p'-q,\lambda'}\, b^*_{p',\lambda'_0} + b_{p'-q,\lambda'}\, a_{p',\lambda'_0} \right] . \quad (3\text{-}6)$$

Here Ω is the normalization volume and K describes the scattering,

$$p, \lambda_0 \to p+q, \lambda ,$$

$$p', \lambda'_0 \to p'-q, \lambda' .$$

The final term V' in Eq. (3-1) represents the connections to the approximate Sawada Hamiltonian,

$$H_s = \mathbb{K} + \mathbb{K}_0 + V_s . \quad (3\text{-}7)$$

We shall return later to an estimate of the importance of V'.

Finally we adopt the Sawada-Wentzel commutation relations

$$\left[b_{p'+q',\lambda'_1} a_{p',\lambda'},\; b_{p+q,\lambda_1} a_{p,\lambda} \right]_-$$

$$= \left[a^*_{p'+q',\lambda'}\, b^*_{p',\lambda'_1},\; a^*_{p+q,\lambda}\, b^*_{p,\lambda_1} \right]_- = 0,$$

$$\left[b_{p'+q',\lambda'_1} a_{p',\lambda'},\; a^*_{p+q,\lambda}\, b^*_{p,\lambda_1} \right]_- \quad (3\text{-}8)$$

$$= \delta_{q',-q}\, \delta_{p',p+q}\, \delta_{\lambda'_1,\lambda_1}\, \delta_{\lambda',\lambda} .$$

When the commutation rules (3-8) are used, the physical interpretation of the Hamiltonian (3-7) is as follows. First, $a^*_{p+q,\lambda}\, b^*_{p,\lambda_1}$ describes the excitation of a nucleon from the state (p, λ_1) to the state $(p+q, \lambda)$; this leaves a "hole" at (p, λ_1). Thus a "pair" is created. The commutation rules (3-8) prescribe that the particle is eventually returned to its original hole (with no virtual scatterings having occurred in the meantime). Thus a particle and a hole are always associated with one another. Such interactions, of course, do not exhaust all possibilities in the complete Hamiltonian (3-1). Consequently, the Sawada Hamiltonian is only approximate. (This approximation has been discussed and used several times previously.[14, 15]) For the present we consider only H_s, neglecting V'. The extra contributions to the energy arising from V' have been called "cluster corrections" by Brueckner and his collaborators. (Corrections to the approximate commutation relations (3-8) are also included in V'.) Our Hamiltonian H_s is considerably more complicated than that of Sawada et al.[10] These authors considered only the Coulomb interactions in a degenerate electron gas. The spin degeneracy of the electrons was trivial in their case, since the Coulomb potential is spin-independent. Also, the use of the K matrix in Eq. (3-6), rather than the matrix elements of a local potential, adds analytic complexities to the eigenvalue equation. As we shall see, however, these difficulties are not insurmountable. Our eigenvalue problem is actually quite similar to that of Sawada et al.[10] We shall find, as did Sawada et al., that the eigenstates of H_s fall into two classes--those corresponding to single-particle excitation and those corresponding to hydrodynamic modes. Sawada et al.[10] classified the hydrodynamic modes as "damped" or "undamped." We feel that this distinction is artificial, as all the hydrodynamic modes are expected to be damped. That is, we anticipate that the eigenstates of H and H_s may bear little resemblance to each other. Expressed differently, a many-particle system is expected to be "ergodic" in the sense that a simple cooperative motion persists for only a limited time. The importance of H_s is thus not that it may give one information concerning the eigenstates of nuclei. Rather, we must think of a time-dependent process by which we excite an eigenstate of H_s at a given time. In calling H_s a "good approximation" to H, we mean that the eigenmodes of H_s do not decay in less than several oscillations. If τ is the time for decay, then $\hbar/\tau$ should be small compared with the energy resolution with which we are studying the eigenmode of H_s.

From the above discussion, it seems clear that we must solve a time-dependent problem, rather than one involving stationary states. Thus we use scattering theory. Let us suppose the excitations to be started impulsively by some external means, such as a particle striking our nuclear matter. Let $\chi(t)$ describe the nuclear matter in its ground state plus the wave packet of the bombarding particle. Also, let us suppose this extra particle interacts only for a short time at $t = 0$. Then the complete wave function for the system is

$$\underline{\Psi} = \chi(t) + \underline{\Psi}_s(t) . \tag{3-9}$$

Here $\underline{\Psi}_s(t)$ represents the effect of the external disturbance on the medium. It arises from a term H_{int} which, with the nuclear matter, describes the interaction of external particles.

Since we are considering a transient problem, the boundary conditions are important. To formulate these, we define

$$D(t) \equiv H_{int}\, \chi(t),$$

with

$$D(t) = \int_{i\eta-\infty}^{i\eta+\infty} dE\, e^{-iEt}\overline{D}(E) . \tag{3-10}$$

For present purposes, we set $D(t) = 0$ for $t < 0$, so that $\overline{D}$ is analytic for $\mathrm{Im}(E) \geq 0$. The "scattered wave" in Eq. (3-9) is the particular solution of the "inhomogeneous Schrödinger equation",

$$\left[i \frac{\partial}{\partial t} - H \right] \underline{\Psi}_s = D(t) . \tag{3-11}$$

We have assumed here that H_{int} may be treated as a small perturbation. Using Eq. (3-10), is then found to be

$$\underline{\Psi}_s(t) = \int_{i\eta-\infty}^{i\eta+\infty} dE\, \frac{e^{-iEt}}{E-H}\, \overline{D}(E) . \tag{3-12}$$

Simple considerations of causality require $\underline{\Psi}_s = 0$ for $t < 0$. Thus, on the contour of integration, $\mathrm{Im}(E)$ must be sufficiently great that the singularities of $(E-H)^{-1}$ lie <u>below</u> the contour. In general this puts a lower limit on η.

First, let us replace H by H_s in Eq. (3-12). This equation then describes a system with resonant eigenmodes which is excited by a transient impulse at $t = 0$. The poles of $(E-H_s)^{-1}$ give the eigenfrequencies of these modes. Now, with the actual Hamiltonian H, rather than H_s, let us choose H_{int} so as to give an initial excitation that corresponds closely to one or a few of the eigenstates of H_s. If H_s is a "good approximation" to H, in the sense used above, the initial excitation will be damped sufficiently slowly that the eigenmodes of H_s may be observed before they decay.

Sawada et al. have stated that a hydrodynamic mode will be strongly damped when its energy lies in the continuum of the spectrum of single-particle excitations. Although true, this is not a necessary condition for damping, since the single-particle excitation spectrum may be extended indefinitely by exciting two, three, etc. particles.

To make these statements more precise, we define

$$a \equiv E - H_s , \tag{3-13}$$

and set

$$\frac{1}{E - H} = \Omega \frac{1}{a} ,$$

so that we have

$$\Omega = 1 + \frac{1}{a-V'} V' . \tag{3-14}$$

We note that E must be considered as complex for the evaluation of integrals. By a theorem on the manipulation of such quantities,[16] we may set

$$\Omega = 1 + \frac{1}{a-\Delta'} \Delta' + \frac{1}{a-\Delta'} V' , \tag{3-15}$$

with

$$\Delta' = V' \frac{1}{a} V' .$$

Let us use the eigenstates of H_s as the representation for Δ' and keep only the matrix Δ'_0 formed from the diagonal elements of Δ'. Then the matrix,

$$\Omega_0 = 1 + \frac{1}{a-\Delta'_0} \Delta'_0 , \tag{3-16}$$

is diagonal in the eigenstates of H_s. Thus,

$$\overline{\Psi}_s = \int dE \left[\Omega_0 \frac{1}{a}\right] e^{-i Et}\overline{D}(E) = \int dE \frac{e^{-i Et}}{E - H_s - \Delta'_0} \overline{D}(E) \tag{3-17}$$

describes the propagating eigenmodes of H_s. Since Δ'_0 has a negative imaginary part, these modes are expected to be damped. The damping occurs as the original modes share their energy with other degrees of freedom. This represents the tendency of the system to be ergodic.[17] We may extend the calculation of Δ'_0 to higher orders[16] by taking the diagonal matrix elements of

$$\Delta' = V' \frac{1}{a} V' + V' \frac{1}{a} V' \frac{1}{a} V' + \ldots \quad . \tag{3-18}$$

There are delicate questions involved in the evaluation of the level-shift matrices K, and correspondingly in conveniently ordering the evaluation of the effects of V'. This is discussed to some extend in Section 7. A complete discussion of the matrices K lies outside our present scope, however, since we are emphasizing here the hydrodynamic properties of nuclei for given K. We hope to return in a later publication to questions related to possible discontinuities in K (i.e. energy gaps[18] in the single-particle spectrum). Applications made in this paper assume that the quantities K are continuous, as has been implied in the work of Brueckner and his collaborators.[8]

4. The Eigenvalue Problem for H_s

The commutation relations (3-8) for the pair variables lead to the following commutators for the interaction energy:

$$\left[a^*_{p-q,\lambda}\, b^*_{p,\lambda_0},\, V_s\right]_- = \frac{-1}{\Omega} \sum_{p',\lambda',\lambda'_0} (p,\lambda_0;p'-q,\lambda'\,|K|\,p-q,\lambda;p',\lambda'_0) \times \left[a^*_{p'-q,\lambda'}\, b^*_{p',\lambda'_0} + b_{p'-q,\lambda'}\, a_{p',\lambda'_0}\right]$$

$$\left[b_{p-q,\lambda}\, a_{p,\lambda_0},\, V_s\right]_- = \frac{1}{\Omega} \sum_{p',\lambda',\lambda'_0} (p,\lambda_0;p'-q,\lambda'\,|K|\,p-q,\lambda;p',\lambda'_0) \times \left[a^*_{p'-q,\lambda'}\, b^*_{p',\lambda'_0} + b_{p'-q,\lambda'}\, a_{p',\lambda'_0}\right] \qquad (4\text{-}1)$$

$$\left[a^*_{p-q,\lambda}\, b^*_{p,\lambda_0},\, K\right]_- = \left[E_p - E_{p-q}\right] a^*_{p-q,\lambda}\, b^*_{p,\lambda_0}$$

$$\left[b_{p-q,\lambda}\, a_{p,\lambda_0},\, K\right]_- = \left[E_p - E_{p-q}\right] b_{p-q,\lambda}\, a_{p,\lambda_0}$$

To obtain the collective eigenfunctions and eigenvalues of H_s, we follow closely the procedure of Sawada, Brueckner, Fukuda, and Brout.[10]

Let Ψ_0 be the ground state of H_s, and E_0 the corresponding energy eigenvalue. Similarly, Ψ_q and $E = E_0 + \Delta(q)$ are the wave function and energy for an excited state. Then we have

$$(K + V_s)\,\Psi_q = (E_0 + \Delta)\Psi_q\,,$$
$$(K + V_s)\,\Psi_0 = E_0\Psi_0\,. \qquad (4\text{-}2)$$

We make the "ansatz"[10] for the operator A^*_q, which creates the collective excitation

$$\Psi_q = A^*_q\,\Psi_0\,, \qquad (4\text{-}3)$$

$$A^*_q \equiv \sum_{p,\lambda,\lambda_0} \left[G^{(+)}_{p,q;-\lambda,-\lambda_0}\, a^*_{p-q,\lambda}\, b^*_{p,\lambda_0} - G^{(-)}_{p,q;-\lambda,-\lambda_0}\, b_{p-q,\lambda}\, a_{p,\lambda_0}\right]. \qquad (4\text{-}4)$$

The quantities G are numerical coefficients to be evaluated in a way to be described below. We must, of course, consider $G^{+}_{p,q}$ to vanish except for $p < p_F$, $|p-q| > p_F$, and $G^{(-)}_{p,q}$ to vanish except for $p > p_F$, $|p-q| < p_F$. The symbol "$(-\lambda)$," etc., in Eq. (4-4) represents the time-reversed state of "λ", etc., for reasons that will soon become apparent. From Eq. (4-2) one may readily show

$$\left[K + V_s, A^*_q\right] = \Delta A^*_q . \tag{4-5}$$

Equation (4-4) is next substituted into Eq. (4-5) and the commutator is evaluated with the help of Eq. (4-1). On equating to zero the coefficients of the annihilation and creation operators, one obtains two sets of equations

$$\left[\Delta - L_p\right] G^{(+)}_{p,q;-\lambda,-\lambda_0} = \frac{1}{\Omega} \sum_{p',\lambda',-\lambda'_0} (p',\lambda'_0;p-q,\lambda\,|K|\,p'-q,\lambda';p,\lambda_0) \times \left[G^{(+)}_{p',q;-\lambda',-\lambda'_0} + G^{(-)}_{p',q;-\lambda',-\lambda'_0}\right], \tag{4-6}$$

$$\left[\Delta - L_p\right] G^{(-)}_{p,q;-\lambda,-\lambda_0} = \frac{-1}{\Omega} \sum_{p',\lambda',\lambda'_0} (p',\lambda'_0;p-q,\lambda\,|K|\,p'-q,\lambda';p,\lambda_0) \times \left[G^{(+)}_{p',q;-\lambda'_0-\lambda'_0} + G^{(-)}_{p',q;-\lambda,-\lambda'_0}\right],$$

$$L_p \equiv E_{p-q} - E_p . \tag{4-7}$$

To simplify these expressions, we use time-reversal and parity invariance:

$$\begin{aligned}(p',\lambda'_0;p-q,\lambda\,|K|\,p'-q,\lambda';p,\lambda_0) &= (-p,-\lambda_0;-p'+q,-\lambda'\,|K|\,-p+q,-\lambda;-p',-\lambda'_0)\\ &= (p,-\lambda_0;p'-q,-\lambda'\,|K|\,p-q,-\lambda;p',-\lambda'_0) .\end{aligned}$$

This allows (4-6) to be rewritten as

$$\left[\Delta - L_p\right] G^{(+)}_{p,q;\lambda,\lambda_0} = \frac{1}{\Omega} \sum_{p',\lambda',\lambda'_0} (p,\lambda_0;p'-q,\lambda'\,|K|\,p-q,\lambda;p',\lambda'_0) \times \left[G^{(+)}_{p',q;\lambda',\lambda'_0} + G^{(-)}_{p',q;\lambda',\lambda'_0}\right],$$

$$\left[\Delta - L_p\right] G^{(-)}_{p,q;\lambda,\lambda_0} = \frac{-1}{\Omega} \sum_{p',\lambda',\lambda'_0} (p,\lambda_0;p'-q,\lambda'\,|K|\,p-q,\lambda;p',\lambda'_0) \times \left[G^{(+)}_{p',q;\lambda',\lambda'_0} + G^{(-)}_{p',q;\lambda',\lambda'_0}\right]. \tag{4-8}$$

The two Eqs. (4-8) are the eigenvalue equations which determine $\Delta, G^{(+)}$ and $G^{(-)}$. Before we proceed to the actual solution of the eigenvalue problem, it is interesting to note that our solution leads to boson commutation rules for the operators that create and destroy the collective modes,

$$\left[A_q, A^*_q\right]_- = \delta_{q,q'}, \quad \left[A_q, A_{q'}\right]_- = 0 .$$

To obtain a practical means of solving Eqs. (4-8) we will assume that K can be approximated as follows (this method can be easily generalized if K is piecewise continuous in several domains):

$$K = \sum_m Cm \left\{p, q\right\}^m \left\{p'\right\}^m , \tag{4-9}$$

where $\left\{p'\right\}^m$ and $\left\{p, q\right\}^m$ are some finite set of functions. Next we introduce

$$M^{(m)(\pm)} \equiv \sum_{p'}^{(\pm)} \left\{p'\right\}^m G_{p', q}, \tag{4-10}$$

where

$$\Sigma^{(+)} \equiv \frac{1}{\Omega} \sum_{\substack{p' < p_F \\ |p' - q| > p_F}}$$

$$\Sigma^{(-)} \equiv \frac{1}{\Omega} \sum_{\substack{p' > p_F \\ p' - q < p_F}} \tag{4-11}$$

The quantities M and G in (4-10) are considered to be column matrices in the index λ, while the coefficients C_m in (4-9) are square matrices.

Equations (4-8) are then

$$G^{(\pm)}_{p, q} = \pm \frac{1}{\left[\Delta - L_p\right]} \sum_m C_m \left\{p, q\right\}^m \left[M^{(m)(+)} + M^{(m)(-)}\right] . \tag{4-12}$$

Multiplying by $\left\{p\right\}^{m'}$ and summing our p gives

$$M^{(m')(\pm)} = \pm \sum_m C_m \left[\sum_p \frac{\left\{p\right\}^{m'} \left\{p, q\right\}^m}{\Delta - L_p}\right] \left[M^{(m)+} + M^{(m)(-)}\right]. \tag{4-13}$$

This set of equations defines the eigenvalue Δ.

We are finally left with the problem of interpretation--that is, relating our quantum-mechanical calculation to the classical hydrodynamic arguments of Section 2. For this purpose we must form wave packets of the eigenstates of H_s and then calculate the expectation value of the nucleon density operator for nucleons of type λ:

$$\underset{\sim}{n}_\lambda = \sum_{\substack{k', k \\ (\text{all } k, k')}} a^*_{k', \lambda} a_{k, \lambda} \frac{e}{\Omega}^{-i(\underset{\sim}{k}' - \underset{\sim}{k}) \cdot \underset{\sim}{X}} . \tag{4-14}$$

In the Sawada approximation, this is to be rewritten as

$$\underset{\sim}{n}_\lambda = \sum_q \sum_p \left[a^*_{p-q, \lambda} b^*_{p, \lambda} + b_{p-q, \lambda} a_{p, \lambda} \right] \frac{e}{\Omega}^{i \underset{\sim}{q} \cdot \underset{\sim}{x}} . \tag{4-15}$$

The complete wave function is

$$\Psi = C_0 \Psi_0 \, e^{-iE_0 t} + C_1 \sum_q \mathcal{A}(q) \, A^*_q \Psi_0 \, e^{-i(E_0 + \Delta)t} , \tag{4-16}$$

when we use Eqs. (4-3) and (4-4). Here C_0 and C_1 are constants and $\mathcal{A}(q)$ is the wave-packet amplitude. Equation (4-16) may easily be generalized to states involving the excitation of several hydrodynamics quanta, in which case our arguments are not changed.

The expectation value

$$n_\lambda = (\Psi, \underset{\sim}{n} \Psi) \tag{4-17}$$

contains constant and time-dependent terms:

$$n_\lambda = n_{\lambda_0} + n'_\lambda(t) . \tag{4-18}$$

The time-dependent term $n'_\lambda(t)$ describes the hydrodynamic motion and the average density. From (4-17) we obtain

$$n'_\lambda(t) = \text{Re} \left\{ C^*_0 C_1 \sum_q \mathcal{A}(q) \frac{e}{\Omega}^{i(\underset{\sim}{q} \cdot \underset{\sim}{x} + \Delta t)} \times \sum_p \left[G^{(+)}_{p, q; -\lambda, -\lambda} + G^{(-)}_{p, -q; -\lambda, -\lambda} \right] \right\} . \tag{4-19}$$

Here Re$\{\ldots\}$ means "real part of $\{\ldots\}$."

For small q, we shall show later that for a class of K's we have $\Delta = aq$, where a is a constant. Thus, $n_{\lambda'}$ satisfies the wave equation

$$\left[\frac{\partial^2}{\partial t^2} - a^2 \nabla^2\right] n_{\lambda'} = 0 . \tag{4-20}$$

The wave equation for the various "components" "λ" are coupled by the λ dependence of the G's in Eq. (4-19). Having established the existence of Eq. (4-20) we may apply the classical consideration of Section 2. However, we must first show that the damping of the wave (which is neglected in Eq. (4-20)) is not important during the time interval in question.

5. First Example

Studies of nuclear structure and scattering indicate that the following is a reasonable approximation to the K-matrix, as obtained by Brueckner et al.:[8,19]

$$K = \delta_{\lambda'_0 \lambda'} \, \delta_{\lambda_0, \lambda} \, f_{\lambda, \lambda'} \, V_0(q) \left[1 - \alpha(p^2 + p'^2 - 2\underset{\sim}{p}\cdot\underset{\sim}{p}')\right] . \tag{5-1}$$

The function $V_0(q)$ is discussed below. The quantities $f_{\lambda,\lambda'}$ are a set of constants defined by the following notation:

$$\begin{aligned} \lambda &= 1 \quad \text{corresponds to } (P\uparrow) \\ \lambda &= 2 \quad \text{corresponds to } (P\downarrow) \\ \lambda &= 3 \quad \text{corresponds to } (N\uparrow) \\ \lambda &= 4 \quad \text{corresponds to } (N\downarrow) , \end{aligned} \tag{5-2}$$

where (P↑) means a proton with spin "up," etc. The physical significance of this form for the K matrix is that we have four interacting "fluids," corresponding to the four systems (5-2). A particle in one fluid always remains in it, since spin and isotopic spin flip have been left out.

We further simplify our problem by considering q to be very small. Then p^2 and p'^2 in (5-1) may each be set equal to p_F^2. [Referring to the expression (4-8) one may easily convince himself that this is valid.] The assumption that q is small restricts us to disturbances with wave numbers q small compared with p_F. A final simplification in this section will be the neglect of the $\underset{\sim}{p}\cdot\underset{\sim}{p}'$ term in Eq. (5-1). (In the next example this term will be considered.)

It is convenient to define

$$V(q) \equiv V_0(q)\left[1 - 2ap_F^2\right] = \frac{4\pi g^2}{\mu^2 + q^2}. \tag{5-3}$$

Here μ is the pion mass and g is a Yukawa coupling constant. The particular form of (5-3) is actually unimportant for our purposes, since only V(0), a constant, is significant in the approximation that q is small. Equation (5-3) represents the strength of interaction of two particles on the Fermi surface.

The quantities $f_{\lambda\lambda'}$ are specified as follows in terms of three dimensionless constants of order unity:[20]

$$f_{11} = f_{22} = f_{33} = f_{44} = c > 0,$$

$$f_{13} = f_{31} = f_{24} = f_{42} = a < 0,$$

$$f_{12} = f_{21} = f_{34} = f_{43} = f_{14} = f_{41} = f_{23} = f_{32} = b < 0. \tag{5-4}$$

The terms with c correspond to repulsive forces, while those with a and b correspond to attractive forces.[19] The choice of (5-4) corresponds roughly to actual nuclear forces if we consider $|a|$, c, and $|b|$ to be comparable in magnitude.

With this choice for K, the Brueckner ground-state energy [Eqs. (3-2) and (3-3)] is

$$K = A\left\{\frac{3}{5}\,\frac{p_F^2}{2M^*} + \frac{1}{8}(a + c + 2b)\, n_0 V_0(0)\left(1 - \frac{6}{5}\, a\, p_F^2\right)\right\}, \tag{5-5}$$

where A is the nuclear mass number and n_0 is the particle density. Since K_0 must be negative, we have

$$c + a + 2b < 0\,. \tag{5-6}$$

Because of the $\delta_{\lambda_0\lambda}\,\delta_{\lambda'_0\lambda'}$ in Eq. (5-1), the G's of Eq. (4-4) have the form

$$G^{(\pm)}_{p,q;\lambda,\lambda_0} = \delta_{\lambda,\lambda_0}\, G^{(\pm)}_{p,q;\lambda}\,. \tag{5-7}$$

We may now define

$$T^{\pm}_{\lambda} \equiv \sum_p{}^{\pm}\, G^{\pm}_{p,q;\lambda}, \tag{5-8}$$

and

$$T_\lambda \equiv T_\lambda^{(+)} + T_\lambda^{(-)} \quad . \tag{5-8}$$

Equations (4-8) now become

$$G^{\pm}_{p,q;\lambda} = \pm \sum_{\lambda'} f_{\lambda\lambda'} \frac{V}{\Delta - L_p} T_{\lambda'} \quad . \tag{5-9}$$

If both sides of this equation are summed over p, the result is

$$T_\lambda^{\pm} = \pm \sum_{\lambda'} f_{\lambda\lambda'} V N_0^{\pm} T_{\lambda'}, \tag{5-10}$$

where

$$N_0^{\pm} \equiv \Sigma^{\pm} \frac{1}{\Delta - L_p} \quad . \tag{5-11}$$

Finally, the two Eqs. (5-10) may be added to give

$$T_\lambda = \sum_{\lambda'} f_{\lambda\lambda'} V N_0 T_{\lambda'} , \tag{5-12}$$

with

$$N_0 = N_0^{+} - N_0^{-} \quad . \tag{5-13}$$

Equation (5-12) is the eigenvalue equation, which determines Δ, the $T_{\lambda'}$s, and thus also the G's in Eq. (5-9).

To evaluate N_0, we continue to make the approximation for small q in the pair excitation energy (and suppose that the quantities K are continuous functions of p, p', and q near the Fermi surface),

$$L_p \simeq - \frac{p \cdot q}{M^*} , \tag{5-14}$$

where M^* is the effective mass of a nucleon at the Fermi surface.[21] The quantity N_0 may be transformed into the intergral,

$$N_0 = (-)\, 2 \, \frac{(2\pi)^{-3}}{M^*} \int_{\substack{p < p_F \\ |p-q| > p_F}} \frac{d^3p \; p \cdot q}{\left[\Delta^2 - \left(\frac{p \cdot q}{M^*}\right)^2\right]} , \tag{5-15}$$

and this may be written as

$$N_0 = -\frac{3}{2}\frac{n_0}{S\epsilon_F}\frac{1}{Q} , \qquad (5\text{-}16)$$

where n_0 is the nucleon density and S is the number of nucleon degrees of freedom ($S = 4$ in our case). We also have

$$\epsilon_F = \frac{p_F^2}{2M^*}$$

and

$$\frac{4\pi}{3} p_F^3 = (2\pi)^3 \frac{N_0}{S} . \qquad (5\text{-}17)$$

The quantity Q is a function of

$$A \equiv \frac{M^*\Delta}{p_F q} , \qquad (5\text{-}18)$$

the ratio of the eigenvalue Δ to the maximum energy for exciting a pair of momentum q.

When the eigenvalue Eq. (5-12) is satisfied by $A > 1$, the integrand (5-15) is nonsingular and evaluation is straightforward:

$$\frac{1}{Q} = 1 - \frac{A}{2}\ell n\left[\frac{A+1}{A-1}\right] . \qquad (5\text{-}19)$$

When no solution to the eigenvalue problem exists for $A \geq 1$, we invoke the considerations of Section 3, where we concluded that Δ must have a positive imaginary part. This condition defines how the singularity in Eq. (5-15) is to be treated; to be specific, it defines the phase of the logarithm in Eq. (5-19). Accordingly we write

$$A + 1 = |A + 1|\, e^{i\alpha} ,$$
$$A - 1 = |A - 1|\, e^{i\pi - i\epsilon} ,$$

where $\alpha, \epsilon > 0$.

Then we have

$$\frac{1}{Q} = 1 - \frac{A}{2}\ell n\left[\frac{|1 + A|}{|1 - A|}\right] + i\pi\frac{A}{2} - i(\alpha - \epsilon)\frac{A}{2} . \qquad (5\text{-}20)$$

In either case the eigenvalue equation (5-12) may be written as

$$Q(A)T_{\lambda} = -\left[\sum_{\lambda'=1}^{4} f_{\lambda\lambda'}\, T_{\lambda'}\right]\left[\frac{3}{2}\ \frac{n_0 V}{\epsilon_F S}\right] . \tag{5-21}$$

The function $1/Q$ is exhibited as a function of A for $1 < A < \infty$ in Fig. 3. It is seen that Q takes on all negative values in this interval. Consequently, there is always one (and only one) value of A with $A > 1$ if the eigenvalue equation (5-21) gives a negative Q. For a single component ($S = 1$) this is true if the forces are repulsive, i.e., $V(0) > 1$. For positive Q there exists no solution with real A--that is, no stable hydrodynamic solution.[22]

Since Q is necessarily real, the imaginary part of the right side of Eq. (5-20) is identically zero. A brief analysis of this equation leads directly to the conclusion that the imaginary part of A, and thus Δ, is positive. This implies that if for the solution of the eigenvalue equation any of the modes corresponds to a positive Q, this mode is not only unstable but also its amplitude increases exponentially with time. Such a situation is of course unphysical for a stable medium, and if the theory leads to exponentially growing waves, this must be ascribed to an improper treatment of the state of the medium described by the Hamiltonian K_0.

For small q, the function V(q) of Eq. (5-3) may be replaced by $V(0) = 4\pi g^2/\mu^2$. The eigenvalue problem, Eq. (5-21), now yields a value for Q(or A) which is independent of q. From the definition of A, Eq. (5-18), we conclude that only the ratio of Δ to q is determined, i.e.,

$$\Delta/q = a .$$

This result was the basis of the previous discussions of nuclear hydrodynamics in Sections 2 and 4.

The actual eigenvalues of Q [Eq. (5-21)], which follow from the coefficients $f_{\lambda\lambda'}$ assumed in Eq. (5-4), are

$$-\frac{Q}{\left[\frac{3}{8}\frac{n_0 V}{\epsilon_f}\right]} = \begin{cases} c - a & \text{(Solution 1)} \\ c - a & \text{(Solution 2)} \\ c + a - 2b & \text{(Solution 3)} \\ c + a + 2b & \text{(Solution 4)} \end{cases}$$

The corresponding T_λ's are

$$T = T_1 \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} \begin{matrix} (P\uparrow) \\ (P\downarrow) \\ (N\uparrow) \\ (N\downarrow) \end{matrix} \qquad \text{(Solution 1)}$$

$$T = T_1 \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} \qquad \text{(Solution 2)}$$

$$T = T_1 \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \qquad \text{(Solution 3)}$$

$$T = T_1 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \qquad \text{(Solution 4)} . \qquad (5\text{-}23)$$

On comparing Eqs. (5-21) and (5-9), we see that $G^{\pm}_{p,q;\lambda}$ is equal to T_λ times a quantity that is independent of λ. According to Eq. (4-17), therefore, the T_λ's are essentially the normal hydrodynamic amplitudes:

$$n'_1 : n'_2 : n'_3 : n'_4 : = T_1 : T_2 : T_3 : T_4 . \qquad (5\text{-}24)$$

By Eq. (5-23) we see that Solution (4) corresponds to a simple compressional mode, since the four fluid components move together. According to the saturation conditions, Eq. (5-6), this eigenmode has a position Q and is unstable, and moreover the waves increase rather than decrease in amplitude with time. We are forced to conclude that the ground state of the medium is not adequately described by our approximate Hamiltonian K_0, Eq. (3-2). The perturbation by V's leads to large, unstable variation in the density of the medium. We may note here that this difficulty is not removed by the more precise treatment of the K matrix given in the next section. Thus at the very least, this may imply the need to include the effect of V's in the evaluation of the ground-state energy. Otherwise certain radical revisions of the K matrix may be necessary to correct this difficulty.

On the other hand, the degenerate Modes 1 and 2 are stable in the Sawada approximation because Eqs. (5-4) imply $c - a > 0$. Solution 1 is a Goldhaber-Teller[3] mode, for according to Eqs. (5-23) and (5-24), the neutron and proton fluids move 180° out of phase. Estimates of a, b, and c indicate that Solution 3 is also expected to be stable, i.e., we anticipated $c + a - 2b > 0$. For this solution, nucleons with spin "up" move out of phase with those having spin "down."

For $A^2 >> 1$ the solution of Eq. (5-19) is $Q = -3A^2$ and the eigenvalue solutions (5-22) are

$$\Delta^2 = \frac{1}{4}\left[\frac{4\pi n_0 g^2}{M^*}\right]\left(q/\mu\right)^2 \begin{cases} (c-a) & \text{(Solution 1)} \\ (c-a) & \text{(Solution 2)} \\ (c+a-2b) & \text{(Solution 3)}. \end{cases} \quad (5\text{-}25)$$

We may call

$$\omega_p^2 = \frac{4\pi n_0 g^2}{M^*} \quad (5\text{-}26)$$

the nuclear "plasma frequency."

To obtain numerical estimates of the energies of the collective oscillations we utilize the numerical values of Karplus and Watson[19] for V(0) and α. For simplicity we pick the absolute values of a, b, and c to be unity, which makes the three stable modes have the same energy. We choose the following numerical values:

$p_F = \frac{270 \text{ Mev}}{c}$, $n_0 V_0 = 50$ Mev, $\alpha = (2p_F)^{-2}$, $M^* = \frac{2}{3}$ M.

It is a simple matter to solve for A, and we find that the value is practically equal to unity, so that we have

$$\Delta = \frac{p_F q}{M^*} = (0.443\ C)\ q.$$

The value of Δ for a large nucleus can be estimated, by taking

$$q = \frac{\hbar\pi}{R} \quad \text{and } R \sim 10^{-12} \text{ cm, so that}$$

$$\Delta \cong 27 \text{ Mev.}$$

This value is larger than the values associated with the Goldhaber-Teller collective oscillation, but is certainly in qualitative agreement, particularly when one considers the probable importances of boundary conditions for a finite system.

6. A Second Example

We now choose a K matrix that is physically more plausible than that of the last section:

$$K = f\ (p + q,\ p'-q\ |K_0|\ p, p'), \tag{6-1}$$

$$f = 1 + c_1\ \sigma \cdot \sigma' + c_2\ \tau \cdot \tau' + c_3\ \sigma \cdot \sigma'\ \tau \cdot \tau', \tag{6-2}$$

$$K_0 = V_0(q) \left[1 - a_0\ (p^2 + p'^2 - 2\ p \cdot p') \right]. \tag{6-3}$$

Here σ and τ are respectively the spin and isotopic spin operators for a nucleon. The coefficients c_1, c_2, and c_3 are constants which will later be chosen to correspond to actual nuclear forces. Finally, a_0 is a constant.

The Brueckner approximation to the ground-state energy is

$$/K_0 = A \left\{ \frac{3}{5}\ \frac{p_F^2}{2\ M} + \frac{1}{2}\ n_0 V_0(0) \left[1 - a_0\ \overline{(p-p')^2} \right] \right\}. \tag{6-4}$$

Here $\overline{(p-p')^2}$ is an average over the Fermi sphere. Since we have $a_0 (p-p')^2 < 1$ we see that we have

$$V_0(0) < 0\ . \tag{6-5}$$

This is the analog of Eq. (5-6).

To anticipate our conclusions, the K-matrix (6-1) leads to precisely the same eigenmodes as were found in Section 5. In addition, we shall find extra modes which correspond to oscillations of the isotopic spin vector density.

Equation (6-1) is now to be substituted into Eqs. (4-4). Paying careful attention to the order of indices, we obtain the two equations

$$\left[\Delta - L_p\right] G^{\pm}_{p,q} = \pm \frac{1}{\Omega} \sum_{p'} K_0 \Big\{ \langle\langle (G^+_{p',q} + G^-_{p',q}) \rangle\rangle$$

$$+ \epsilon_1 \underset{\sim}{\sigma} \cdot \langle\langle \underset{\sim}{\sigma} (G^+_{p',q} + G^-_{p',q}) \rangle\rangle$$

$$+ \epsilon_2 \underset{\sim}{\tau} \cdot \langle\langle \underset{\sim}{\tau} (G^+_{p',q} + G_{p',q}) \rangle\rangle$$

$$+ \epsilon_3\, \sigma_i\, \tau_j \quad \langle\langle \sigma_i \tau_j (G^+_{p',q} + G_{p',q}) \rangle\rangle \Big\} \quad . \tag{6-6}$$

It is convenient to consider $G^{\pm}_{p,q}$ as rectanglar matrices in spin and i-spin space and omit the λ indices. The symbol $\langle \ldots \rangle$ represents a trace in both spin and i-spin variables. Finally, a sum over repeated vector indices is implied in the last term above.

Multiplying Eqs. (6-6) successively by $1, \underset{\sim}{\sigma}, \underset{\sim}{\tau}$, and $\sigma_i \tau_j$ and forming the trace, we obtain the decoupled equations

$$\left[\Delta - L_p\right] \Gamma_{4p}^{\pm} = \pm \frac{4}{\Omega} \sum_{p'} K_0 \left[\Gamma_{4p'}^{+} + \Gamma_{4p'}^{-}\right],$$

$$\left[\Delta - L_p\right] \underset{\sim}{\Gamma}_{3p}^{\pm} = \pm \frac{4\epsilon_1}{\Omega} \sum_{p'} K_0 \left[\underset{\sim}{\Gamma}_{3p'}^{+} + \underset{\sim}{\Gamma}_{3p'}^{-}\right],$$

$$\left[\Delta - L_p\right] \underset{\sim}{\Gamma}_{1p}^{\pm} = \pm \frac{4\epsilon_2}{\Omega} \sum_{p'} K_0 \left[\underset{\sim}{\Gamma}_{1p'}^{+} + \underset{\sim}{\Gamma}_{1p}^{-}\right],$$

$$\left[\Delta - L_p\right] \left(\Gamma_{2p}^{\pm}\right)_{\ell m} = \pm \frac{4\epsilon_3}{\Omega} \sum_{p'} K_0 \left[\left(\Gamma_{2p'}^{+}\right)_{\ell m} + \left(\Gamma_{2p'}^{-}\right)_{\ell m}\right]. \tag{6-7}$$

We have here introduced the abbreviations

$$\Gamma_{4p}^{\pm} \equiv \langle G_{p,q}^{\pm} \rangle ,$$

$$\underset{\sim}{\Gamma}_{3p}^{\pm} \equiv \langle \underset{\sim}{\sigma}\, G_{p,q}^{\pm} \rangle ,$$

$$\underset{\sim}{\Gamma}_{1p}^{\pm} \equiv \langle \underset{\sim}{\tau}\, G_{p,q}^{\pm} \rangle ,$$

$$\left(\Gamma_{2p}^{\pm}\right)_{\ell m} \equiv \langle \sigma_\ell\, \tau_m\, G_{p,q}^{\pm} \rangle . \tag{6-8}$$

The G's corresponding to these four classes of eigenmodes are

$$G^{\pm}_{p,q}(4) = \frac{1}{4}\Gamma^{\pm}_{4p} ,$$

$$G^{\pm}_{p,q}(3) = \frac{1}{4}\underset{\sim}{\sigma}\cdot\underset{\sim}{\Gamma}^{\pm}_{3p} ,$$

$$G^{\pm}_{p,q}(1) = \frac{1}{4}\underset{\sim}{\tau}\cdot\underset{\sim}{\Gamma}^{\pm}_{1p} ,$$

$$G^{\pm}_{p,q}(2) = \frac{1}{4}\ \sigma_{\ell}\ \tau_{m}\ (\Gamma^{\pm}_{2p})_{\ell m} \quad . \tag{6-9}$$

The numbering of these eigenmodes is chosen to correspond to that of Section 5. At present, however, we have sixteen eigenmodes rather than four, although we have only four different eigenvalue equations. These extra degenerate modes are due to the directional degeneracy in spin and i-spin space. The effect of this is to permit us to set up a nuclear vibration in such a manner that the spin and i-spin vectors oscillate. Thus, we may have "spin waves" and "isotopic spin waves."

The discussion in Section 4 showed that the macroscopic density, n'_{λ} is proportional to the diagonal elements (in λ, the spin, i-spin index) of the G functions. For the excitation of any given eigenmode of collective oscillation, the ratios of density of the components of the medium can be obtained immediately from inspection of the G functions given in Eq. (6-9). Thus for the component, protons with spin up, in eigenmode Number 3, we have

$$n'_{1} \approx (G^{\pm}_{p,q}(3)\ )_{1,1} = \frac{1}{4}\ (\underset{\sim}{\sigma})_{1,1}\cdot\ \underset{\sim}{\Gamma}^{\pm}_{3p} \ .$$

We note that only excitations of the third component of $\Gamma^{\pm}_{3p}$ can lead to density fluctuations, since only σ_3 has diagonal elements. Proceeding in this manner we can obtain the ratios of the densities of the various components in all the eigenmodes, which we list:

$$n_1:n_2:n_3:n_4 = 1:1:(-1):(-1), \qquad \text{Solution 1,}$$

$$n_1:n_2:n_3:n_4 = 1:(-1):(-1):1, \qquad \text{Solution 2,} \tag{6-10}$$

$$n_1:n_2:n_3:n_4 = 1:(-1):1:(-1), \qquad \text{Solution 3,}$$

(6-10 con'd)

$$n_1:n_2:n_3:n_4 = 1:1:1:1, \qquad \text{Solution 4.}$$

These are seen to correspond exactly to the solutions in Section 5.

The solutions (6-11) do not exhaust the possible motions. We may calculate, for instance, the average spin density,

$$\langle \underset{\sim}{\sigma} \rangle = \langle \Psi_q, \sum_{\substack{kk' \\ \lambda\lambda_0}} a^*_{k'_\lambda} a_{k_{\lambda_0}} \underset{\sim}{\sigma}_{\lambda\lambda_0} \frac{e^{-i(k'-k)\cdot r}}{\Omega} \Psi_q \rangle ,$$

in the same manner as the density was calculated in Section 4. This is straightforward and leads to the expression

$$\langle \underset{\sim}{\sigma} \rangle = \mathrm{Re} \left\{ \mathcal{C}_1 \mathcal{C}^*_0 \sum_q a(q) \frac{e^{i(\underset{\sim}{q}\cdot \underset{\sim}{x} + \Delta t)}}{\Omega} \times \sum_p \left[\underset{\sim}{\Gamma}^+_{3p} + \underset{\sim}{\Gamma}^-_{3p} \right] \right\} . \qquad (6\text{-}11)$$

We may now for example excite all three components of $\underset{\sim}{\Gamma}_{3p}$ and phase them to correspond to a rotation of the average spin-vector density, which is then propagated as a spin wave. Similarly, we can obtain the propagation of i-spin waves,[6] and coupled spin and i-spin waves.

We next require the eigenvalues Δ for each of the four Eqs. (6-7). It clearly suffices to calculate any one of these and then insert the appropriate factor $\mathcal{C}_i$ into the expression. For definitions we shall take the first of Eqs. (6-7). Again, q is considered to be very small, so that Eq. (6-3) may be written as

$$K_0 = V_0(q) \left[1 - 2a_0 p_F^2 + 2a_0 \underset{\sim}{p} \cdot \underset{\sim}{p'} \right] ,$$

$$= V(q) \left[1 + 2a \underset{\sim}{p} \cdot \underset{\sim}{p} \right] . \qquad (6\text{-}12)$$

This equation implicitly defines V and a in terms of V_0 and a_0.

Let us now define

$$T^\pm \equiv \sum_{p'}{}^{\pm} \Gamma^\pm_{4p'} ,$$

$$\underset{\sim}{R}^\pm \equiv \sum_{p'}{}^{\pm} \underset{\sim}{p'} \Gamma^\pm_{4p} . \qquad (6\text{-}13)$$

The notation $\Sigma^{\pm}$ is that of Eqs. (4-8). From these we obtain

$$T \equiv T^{+} + R^{-} ,$$
$$\underset{\sim}{R} \equiv \underset{\sim}{T}^{+} + \underset{\sim}{R}^{-} . \qquad (6\text{-}14)$$

With this notation, the first of Eqs. (6-7) becomes

$$\Gamma^{\pm}{}_{4p} = \pm \frac{1}{\Delta - L_p} V\left[T + 2a\underset{\sim}{p} \cdot \underset{\sim}{R}\right]. \qquad (6\text{-}15)$$

Six integrals are needed to solve this equation:

$$N^{\pm}{}_0 \equiv \frac{1}{\Omega} \Sigma^{\pm}_{p} \frac{1}{\Delta - L_p} ,$$
$$\hat{q}\, N^{\pm}{}_1 \equiv \frac{1}{\Omega} \Sigma^{\pm}_{p} \frac{\underset{\sim}{p}}{\Delta - L_p} ,$$
$$\hat{q}\hat{q}\, N^{\pm}{}_2 \equiv \frac{1}{\Omega} \Sigma^{\pm}_{p} \frac{\underset{\sim}{p}\,\underset{\sim}{p}}{\Delta - L_p} . \qquad (6\text{-}16)$$

In terms of these, there will occur

$$N_0 \equiv N_0{}^{+} - N_0{}^{-} ,$$
$$N_1 \equiv N_1{}^{+} - N_1{}^{-} ,$$
$$N_2 \equiv N_2{}^{+} - N_2{}^{-} . \qquad (6\text{-}17)$$

On multiplying Eq. (6-15) by 1 and by $\underset{\sim}{p}$, and in each case summing over $\underset{\sim}{p}$, we find

$$T^{\pm} = \pm V \left[N_0{}^{\pm} T + 2aN_1{}^{\pm} \hat{q} \cdot \underset{\sim}{R}\right],$$
$$\underset{\sim}{R}^{\pm} = \pm V \left[N_1{}^{\pm} T + 2aN_2{}^{\pm} \hat{q} \cdot \underset{\sim}{R}\right] \hat{q} .$$

From these we obtain

$$T = V\left[N_0 T + 2a\, N_1 \hat{q} \cdot \underset{\sim}{R}\right] ,$$
$$\hat{q} \cdot \underset{\sim}{R} = V\left[N_1 T + 2a\, N_2 \hat{q} \cdot \underset{\sim}{R}\right] .$$

Eliminating $\hat{q} \cdot \underset{\sim}{R}$ from these, we find the algebraic equation for Δ:

$$1 = V N_0 + \frac{2a(VN_1)^2}{1 - 2aVN_2} . \qquad (6\text{-}18)$$

We use the notation of Section 5. It was seen there that the only undamped waves (in the present approximation) are those for $A > 1$. In this case we have

$$N_1 = -\frac{M^* \Delta}{q} N_0 ,$$

$$N_2 = -\frac{n_0 M^*}{S} + A^2 p_F^2 N_0 . \qquad (6\text{-}19)$$

N_0 is given by Eqs. (5-16) and (5-19).

We introduce the abbreviation

$$\Gamma = \frac{3 n_0 V}{2 S \epsilon_F} , \qquad (6\text{-}20)$$

and find that Eq. (6-18) reduces to

$$1 = -\frac{\Gamma}{Q} + \frac{2(a p_F^2) \dfrac{\Gamma^2 A^2}{Q^2}}{1 + 2(a p_F^2)\left[\dfrac{A^2 \Gamma}{Q} + \dfrac{1}{3}\Gamma\right]} .$$

This, in turn, can be written in the form

$$\frac{\Gamma}{Q} = \frac{1 + (2 a p_F^2)\, \Gamma/3}{1 + (2 a p_F^2)\left(\dfrac{\Gamma}{3} + A^2\right)} , \qquad (6\text{-}21)$$

which reduces to Eq. (5-22) for $a \to 0$.

In Section 5, it was concluded from an examination of the eigenvalue equation that repulsive forces ($\Gamma > 0$) led to stable collective oscillation and that attractive forces ($\Gamma < 0$) led to inherently unstable (and unphysical) eigenmodes. These conclusions remain valid for repulsive forces; and also for attractive forces for

$$2 a p_F^2 \frac{|\Gamma|}{3} < 1.$$

For typical values of a and Γ such as we are using, these conditions are satisfied, so that the previous conclusions are maintained. The eigenmode corresponding to the compressive wave is unstable and furthermore is exponentially growing. Our improved treatment of the momentum dependence of the K matrix in this section did not change that unsatisfactory aspect of our results. It would require a substantial change in the values of our parameters to make the compressive mode stable. The spin and i-spin eigenmodes will be stable if the constants c_1, c_2, and c_3 are negative. Negative values for these constants are compatible with the low-energy nucleon-nucleon scattering, and we conclude that these modes are stable. Just as in the preceding section, though, we will find that the value of A is very close to unity and thus relatively insensitive to the precise value of the potential. Accordingly we simply proceed with the numerical estimates of Eq. (6-21), using the values of V_0 and a_0 determined by Karplus and Watson.[19] This yields $\Gamma = 0.321$, and we find $A = 1.01$, so that very nearly, just as before,

$$\omega = \frac{p_F q}{M^*} ,$$

$$= (0.441C)q.$$

7. Damping of the Collective Eigenstates

It has been emphasized that the stable collective eigenmodes that appear in the Sawada approximation are only relatively or quasi-stable. Those terms in the Hamiltonian, V', ignored in this approximation cause the damping of these eigenmodes. These eigenmodes are degenerate with respect to two-pair, three-pair, etc., states, and the perturbation V' transforms the ordered collective motion into the noncorrelated motion of two or more pair states. If the Sawada approximation is good this damping will be small.

According to Section 3, the lowest-order damping is given by

$$\Delta_0 = \mathrm{Im}\left\{\left(\Psi_q, V' \frac{1}{a} V' \Psi_q\right)\right\}. \qquad (7\text{-}1)$$

The fact that we are using K matrices for V' does not change this form, since we obtain K matrices from potentials by summing over pairs of terms involving a pair of nucleons. In doing this we do not lose any qualitative features of class of terms. We may write Eq. (7-1) in the form

$$\Delta_0 = -\pi \sum_E \delta(\Delta(q) - E) \left| \Psi_E, V' \Psi_q \right|^2 \rho_E , \qquad (7\text{-}2)$$

where $\Delta(q)$ is the energy of the collective eigenmode, and ρ_E is the density of states, Ψ_E. The simplest state, Ψ_E, degenerate with the collective state Ψ_q is a two-pair state. It can be verified easily that there are two classes of terms in V' for which this matrix element exists. These are

$$V'_1 = \frac{1}{2\Omega} \sum_{\substack{p, p' q \\ \lambda \lambda_0 \\ \lambda' \lambda'_0}} \Big\{ (p + q, \lambda; p'-q, \lambda' |K| p, \lambda_0; p', \lambda'_0) \Big[a^*_{p+q, \lambda} a^*_{p'-q, \lambda'} a_{p, \lambda_0} b^*_{p', \lambda'_0} + a^*_{p+q, \lambda} a^*_{p'-q, \lambda'} b^*_{p, \lambda_0} a_{p', \lambda'_0} \Big] \Big\} ,$$

$$V'_2 = \frac{1}{2\Omega} \sum_{\substack{p, p', q \\ \lambda, \lambda_0 \\ \lambda', \lambda'_0}} \Big\{ (p+q, \lambda; p'-q, \lambda' |K| p, \lambda_0; p', \lambda'_0) \Big[a^*_{p+q, \lambda} b_{p'-q, \lambda'} b^*_{p\lambda_0} b^*_{p'\lambda'_0} + b_{p+q, \lambda} a^*_{p'-q, \lambda'} b^*_{p\lambda_0} b^*_{p', \lambda'_0} \Big] \Big\} . \tag{7-3}$$

The potential V'_1 operating on a collective eigenmode results in the scattering of the excited particle of the collective eigenmode with a particle in the Fermi gas, and the raising of the latter up above the Fermi energy, thus creating two pairs. The potential V'_2 operating on the collective eigenmode can be regarded as the scattering of the hole member of the pair in the collective eigenmode in the Fermi gas, resulting in the creation of two pairs. We represent the two-pair state Ψ_E simply as

$$\Psi_E = a^*_{p_1+q, \lambda_1} b^*_{p_1, \lambda'_1} a^*_{p_2+q_2, \lambda_2} b^*_{p_2, \lambda'_2} \Psi_0 . \tag{7-4}$$

We neglect the fact that we should put in single-pair scattering solutions,[10] since this would be a high-order effect.

Before we proceed with the details of the calculation of Eq. (7-2), it is instructive to establish the connection of our damping term with the classical collision damping in plasmas.[22] Very simply we can interpret the square of the matrix element in Eq. (7-2) in term of an effective nucleon-nucleon cross section, so that roughly we can write

$$\Delta_0 \approx = \frac{1}{2} \sum_{p, p'} \sigma \frac{v}{\Omega} \left| G^+_{p+q, p} \right|^2 , \tag{7-4}$$

where σ is the effective cross section, $\bar{v}$ is the mean velocity of the excited particle, and $G^{+}_{p,q}$ is the usual function appearing in the collective eigenstate. The sums over p, p' are restricted in the usual way by the exclusion principle. Now we have

$$\sum_{p} \left| G^{+}_{p} \right| \approx 1 \quad \text{and} \quad \sum_{p} \approx N(q) ,$$

where $N(q)$ is the number of nucleons at the Fermi surface to depth q. This leads to the result

$$\Delta_0 \approx - \frac{1}{2} \ (n\sigma\bar{v}), \quad n = \frac{N(q)}{\Omega} , \tag{7-5}$$

which is the form of the classical damping term[23] and serves as an interpretation of our calculations here.

Returning to Eq. (7-2), we can now evaluate the matrix element $(\Psi_E, V'\Psi_q)$ using the stated potentials and wave functions. In order to avoid unnecessary complications in our estimate of the damping we neglect the spin dependence and take account of the momentum dependence as in Section 5. As heretofore, we are concerned only with the evaluation for small q, and with these considerations each of the four terms contained in V'_1 and V'_2 yields equal results and our expression for Δ_0 becomes

$$\Delta_0 = - \frac{4\pi V^2 \Omega}{(2\pi)^9} \int \left| G^{+}_{p,+q,q} \right|^2 \delta(q-q_1-q_2)\delta(\Delta(q)-E) \times d^3p_1 \, d^3p_2 \, d^3q_1 \, d^3q_2 \ .$$

From Eq. (5-9), we have

$$G^{+}_{p_1+q,q} = \frac{V}{\Omega} \ \frac{1}{\Delta(q) - \frac{p_1 \cdot q}{M^*}} \ T ,$$

which we will approximate by neglecting the dependence on $\frac{p_1 \cdot q}{M^*}$ in the denominator. Though this leads to an underestimate in Δ_0, our subsequent approximation for T tends to cancel this. The resulting integral is still quite tedious, and we have estimated it with the result for Δ_0 of

$$\Delta_0 = - \frac{4\pi V^2}{(2\pi)^9} \left(\frac{V^2 T^2}{\Omega} \right) \frac{1}{\Delta^2(q)} \ (4\pi^3 M^* p_F{}^3 q^4) . \tag{7-6}$$

The normalization condition that the G functions must satisfy[10] is

$$\sum_{\substack{p<p_F \\ |p+q|>p_F}} \left| G^{+}_{p+q,\,q} \right|^2 - \sum_{\substack{p>p_F \\ |p+q|<p_F}} \left| G^{-}_{p+q,\,p} \right|^2 = 1 .$$

From this one can easily obtain the result (for $q \ll p_F$),

$$\frac{V^2 T^2}{\Omega} = \left\{ \frac{M^{*2}}{(2\pi)^2 q} \left[\frac{2A}{A^2-1} + \ln \frac{A-1}{A+1} \right] \right\}^{-1} . \tag{7-7}$$

In the limit of large A this is

$$\frac{V^2 T^2}{\Omega} = \left\{ \frac{M^{*2}}{(2\pi)^2 q} \frac{4}{3A^3} \right\}^{-1} \tag{7-7a}$$

Since A in our previous numerical estimates was ~ 1, this represents an overestimate in our evaluation of Δ_0; however, our approximation for the function G was an underestimate, and this should at least correct that. Substituting Eq. (7-7a) in Eq. (7-6), we obtain

$$\Delta_0 = - \frac{3}{4} \frac{V^2}{(2\pi)^3} M^* p_F q^3 A. \tag{7-8}$$

The ratio of this with $\Delta = A p_F q / M^*$ is

$$\frac{\Delta_0}{\Delta} = - \frac{3}{4} \frac{V^2}{(2\pi)^3} M^{*2} q^2 = \frac{3}{2} \frac{V^2}{(2\pi)^3} M^{*3} \epsilon_F \left(\frac{q}{p_F} \right)^2 . \tag{7-9}$$

We can very easily compare this to Eq. (7-5), the classical expression for damping. We take for the quantities appearing in Eq. (7-5), the following reasonable estimates:

$$\sigma \approx \frac{M^{*2}(4V)^2}{(2\pi)^2} \left(\frac{\pi q^2}{4\pi p_F^2} \right) ; \quad n \approx \frac{4\pi p_F^2 q}{(2\pi)^3} ; \quad v \approx \frac{p_F}{M^*} .$$

Substituting these values in Eq. (7-5), we obtain

$$\Delta_0 \approx \frac{2}{\pi} \frac{V^2}{(2\pi)^3} M^* p_F q^3 ,$$

which possesses the same form as Eq. (7-8), except for numerical factors.

Substituting our numerical values used in the previous sections into Eq. (7-9), we obtain

$$\frac{\Delta_0}{\Delta} = (-)\ 0.125\ \left(\frac{q}{p_F}\right)^2 .$$

For a large nucleus $(R \sim 10^{-12}\text{cm})$ this becomes

$$\frac{\Delta_0}{\Delta} = (-)\ 6 \cdot 10^{-3} .$$

Thus it appears that the damping is quite small and that the stable Sawada collective states are approximate eigenstates of the total Hamiltonian. Of course, we must emphasize the qualitative nature of this result, particularly as it applies to actual nuclei.

8. Conclusions

We have attempted, first of all, to show how the method of Sawada et al. can be used to provide a detailed description of "macroscopic" hydrodynamic motion. In particular, this may be done for coupled systems of particles (leading to spin waves in our example). The formulation in terms of level-shift K matrices permits a generalization of the method in that the precise definition of the K matrices enters at a separate stage of the calculation. We have suggested that the problem is most properly developed within a time-dependent framework, in which the damping appears as a consequence of the approximations made. In this sense, it is seen to be unimportant to establish a connection with the true eigenstates of the system [the understanding of which is undoubtedly beyond our reach].

We assert that the so-called "single-particle states," whose excitation energy is given by the $L_{p,q}$, must be understood in the same manner. These states also will be damped in time, with a characteristic time (say) τ (p, q).

For $L_{p,q} \gg \dfrac{\hbar}{\tau(p,q)}$ the single-particle energy will lie within a small distance $\hbar/\tau(p,q)$ of the energy of a true eigenstate.

The problem of obtaining the K matrices is not a simple one. We have avoided this in our discussions, using forms suggested by the work of Brueckner and his colleagues. In view of the instability of the hydrodynamic mode obtained here, the possibility of discontinuities in the K matrices at the Fermi surface should be reinvestigated.[18] It also appears that ground-state energy should be re-evaluated by the Sawada method to determine if a higher equilibrium density is implied. We hope to return to these questions in a later publication.

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20. The a used here is not related to the sound speed used previously.

21. $M^* \approx \frac{2}{3}$ M according to Reference 9.

22. Eq. (5-2) and its interpretation as a condition for the stability if collective oscillation in a Fermi gas or liquid have been obtained previously by L. D. Landau, by a simiclassical approach to the problem. L. D. Landau, J. Exptl. Theoret. Phys. (USSR), 32, 59, 1957 Soviet Physics JETP, 5, 101 (1957).

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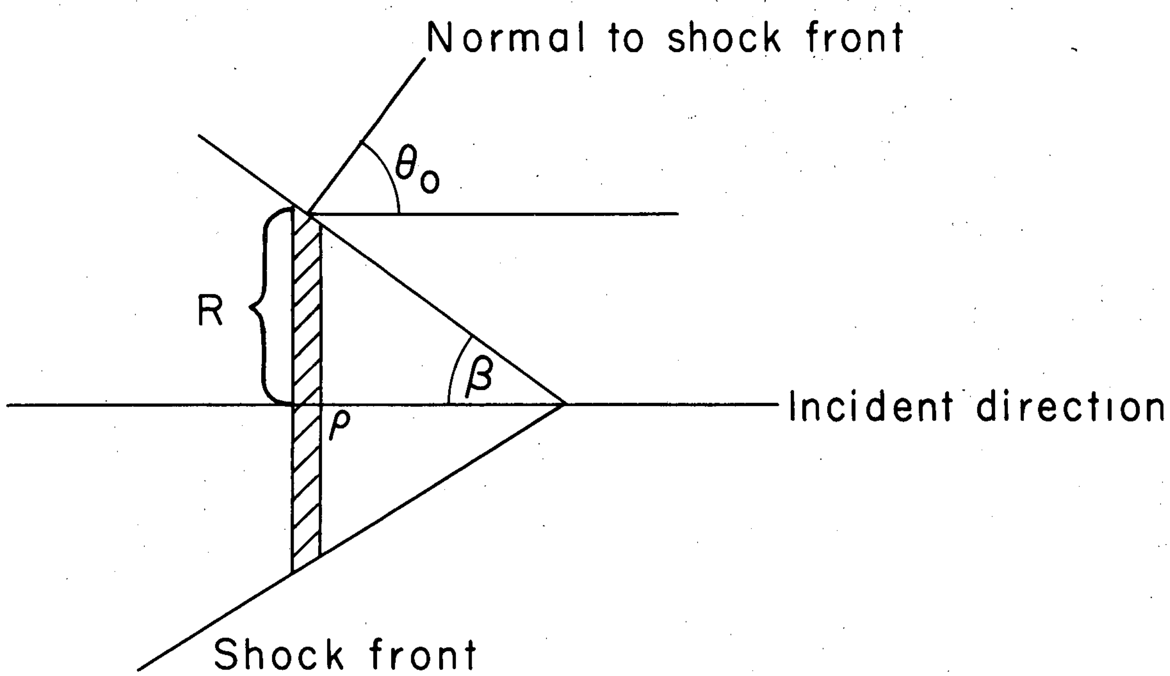

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Fig. 1. Geometry of the shock wave in nuclear matter initiated by a very-high-energy incident particle.

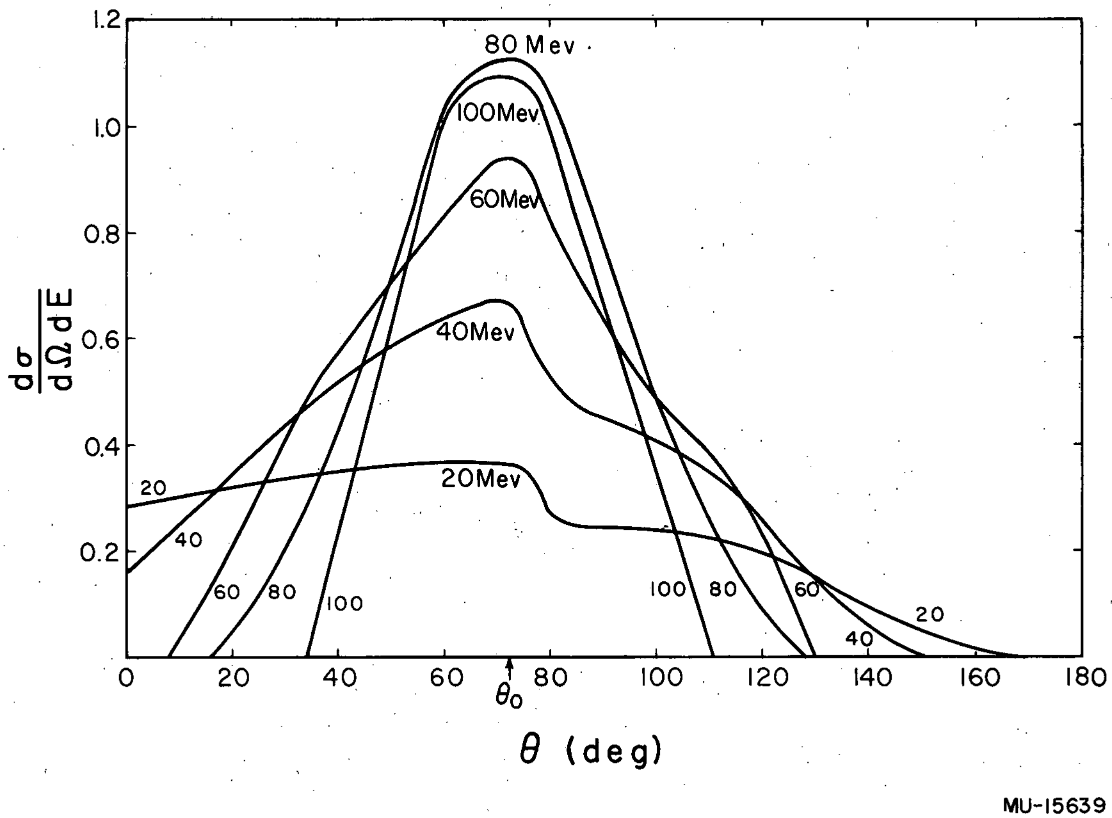

Fig. 2. The angular and energy distributions of nucleons emitted though the excitation of shock waves in nuclei by a very-high-energy incident particle. For the values of the parameters given in the text, the maximum energy for the emitted particles is about 115 Mev. The units of the ordinate are relative.

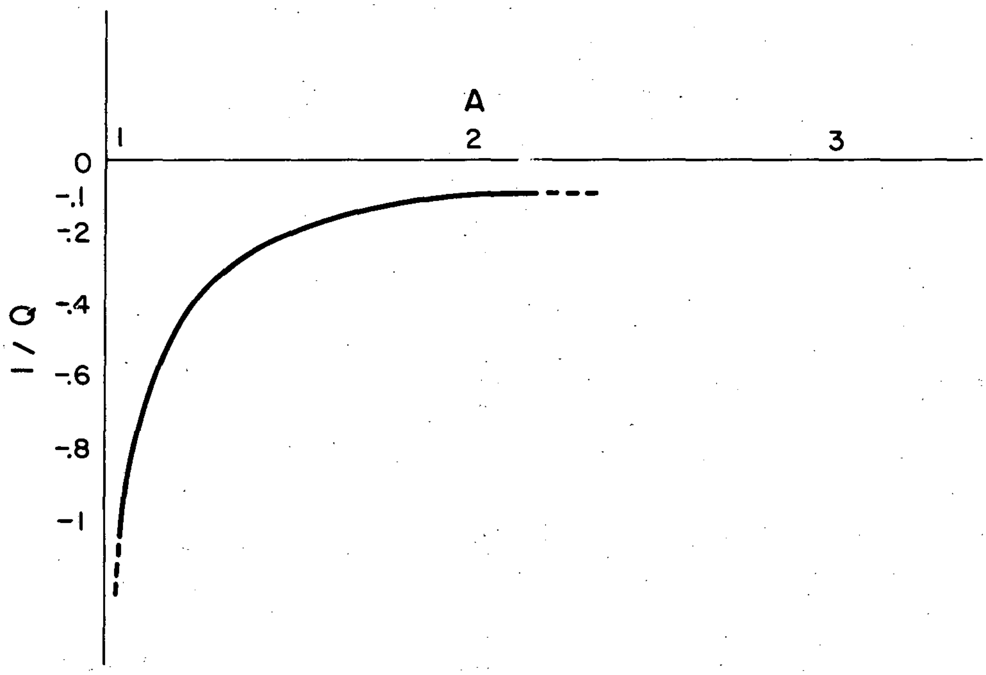

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Fig. 3. Plot of 1/Q as a function of A(Eq. 5-19).