Two-User Multicast Networks with Variable-Length Limited Feedback

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Abstract

We investigate the channel quantization problem for two-user multicast networks where the transmitter is equipped with multiple antennas and either receiver is equipped with only a single antenna. Our goal is to design a global quantizer to minimize the outage probability. It is known that any fixed-length quantizer with a finite-cardinality codebook cannot achieve the same minimum outage probability as the case where all nodes in the network know perfect channel state information (CSI). To achieve the minimum outage probability, we propose a variable-length global quantizer that knows perfect CSI and sends quantized CSI to the transmitter and receivers. With a random infinite-cardinality codebook, we prove that the proposed quantizer is able to achieve the minimum outage probability with a low average feedback rate. Numerical simulations also validate our theoretical analysis.

Keywords

multicast, variable-length quantizer, outage probability, limited feedback

I. Introduction

It is known that using more than one antenna at the transmitters can greatly improve the performance of communication systems. However, the performance depends on the availability of channel state information (CSI) at the transmitters and receivers [1], [2]. Receivers can obtain CSI through training sequences; however, the transmitters must rely on the feedback information from receivers to do so. Additionally, perfect CSI at the transmitters requires an “infinite” number

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of feedback bits, which is unrealistic due to the limitations of feedback links. Therefore, it is more practical to employ quantized CSI to design efficient transmission schemes for wireless networks.

There has been a lot of work on channel quantization in point-to-point multiple antenna systems. An overview of research on limited feedback can be found in [3]. In multiple-input single-output (MISO) systems, a fixed-length quantizer (FLQ) is proposed in [1] to maximize the capacity by applying the beamforming vector at the transmitter. In FLQs, the number of feedback bits per channel state is a fixed positive integer. Compared to the case that all the nodes know CSI perfectly, fixed-length quantization always suffers from some performance loss. On the other hand, [4] proposes a variable-length quantizer (VLQ) to achieve the full-CSI outage probability with a low average feedback rate. VLQs borrow the idea from variable-length coding to allow binary codewords of different lengths to represent different channel states. It has shown in [4] that variable-length quantization does not suffer from performance loss in MISO systems.

In this paper, we study the channel quantization problem in multicast networks with two receivers. We use transmit beamforming and consider the outage probability gap between the proposed quantizer and the full-CSI case. For a FLQ, the standard encoding rule is to choose the codeword “closest” to the channel state. For any finite-cardinality codebook, the outage probability of a FLQ is strictly worse than that of the full-CSI case [4]. To achieve the full-CSI outage probability with a finite average feedback rate, we propose a VLQ with a codebook of infinite cardinality. We expect that in such a VLQ, the codeword covering a larger partition of channel space can be represented by a fewer number of bits. In this way, the average feedback rate can be made finite.

Based on the above analysis, we propose a VLQ in multicast networks that has access to full CSI and sends quantized CSI to the transmitter and receivers via error-free and delay-free feedback links. We consider a random codebook with infinite cardinality that is tractable for analysis [5]. We prove that the outage probability for the VLQ is the same as the full-CSI case. Afterwards, through a derived upper bound on the average feedback rate, we will show that: (i) the average feedback rate is finite and small in the entire range of transmit power; (ii) the average feedback rate will converge to zero when the transmit power approaches infinity or zero. In addition to theoretical analysis, numerical simulations are presented to verify the effectiveness of the proposed VLQ.

The remainder of this paper is organized as follows. In Section II, we describe the system model. In Section III, we depict the proposed VLQ, including its encoding rule and the infinite-cardinality random codebook. In Section IV, we prove that the proposed VLQ achieves the minimum outage probability. An upper bound on the average feedback rate is given in Section V. Numerical simulations are shown in Section VI to validate our theoretical analysis. We draw the conclusions and introduce future work in Section VII. Some technical proofs are provided in the appendices.

Notations: $\top$ represents transpose and $\dagger$ represents conjugate transpose. $\mathtt{C}$ denotes the set of complex numbers and $\mathtt{C}^{m\times n}$ denotes the set of complex vectors or matrices. $\mathtt{CN}\left(a,b\right)$ represents a circulary-symmetric complex Gaussian random variable (r.v.) with mean a and covariance b. $\mathrm{E}\left[\cdot\right]$ denotes the expectation and $\mathrm{Prob}\left\{\cdot\right\}$ denotes the probability. $\mathbb{N}$ is the set consisting of all natural numbers. For any real number x, $\lfloor x\rfloor$ is the largest integer that is less than or equal to x. $\mathbf{1}_{\mathrm{ST}}=\mathbf{1}$ when the logical statement ST is true, and 0 otherwise. Finally, $f_X(\cdot)$ is the probability density function (PDF) for r.v. X.

II. System Model

Consider a multicast network where a transmitter with t antennas ($t\geq 2$) is sending common information to two singe-antenna receivers. The channel vector from the transmitter to receiver m is denoted by $\mathbf{h}_m=\left[h_{m1}\cdots h_{mt}\right]^\top\in\mathtt{C}^{t\times 1}$, where $h_{mn}\simeq\mathtt{CN}(0,1)$ for $m=1,2$, $n=1,\dots,t$. Let $\chi_m=||\mathbf{h}_m||^2$ for $m=1,2$, and $\mathbf{H}=[\mathbf{h}_1\;\mathbf{h}_2]\in\mathtt{C}^{t\times 2}$ represents the entire channel state. At the transmitter, $\mathbf{x}\in\mathcal{X}\triangleq\{\mathbf{x}:\mathbf{x}\in\mathtt{C}^{t\times 1},||\mathbf{x}||^2=1\}$ is employed as the beamforming vector and a scalar symbol $s\in\mathtt{C}$ is sent through t antennas. The received signal at receiver m is

$$y_m=\sqrt{P}\mathbf{x}^\dagger\mathbf{h}_m s+g_m,$$

where P denotes the transmit power and $g_m\simeq\mathtt{CN}(0,1)$ is the additive white Gaussian noise term. We assume $\mathrm{E}\left[|s|^2\right]=1$.

For the multicast network, the maximum achievable rate is $\log_2\left(1+P\min_{m=1,2}\left|\mathbf{x}^\dagger\mathbf{h}_m\right|^2\right)$ [6].[1] Let $\gamma\left(\mathbf{x},\mathbf{H}\right)=\min_{m=1,2}\left|\mathbf{x}^\dagger\mathbf{h}_m\right|^2$, then, for the target data transmission rate ρ, an outage

[1]In this paper, we only consider the channel quantization problem for transmit beamforming. Although the precoding matrix can have higher rank than the beamforming vector, it can be deduced from [6, Theorem 1] and [6, Theorem 2] that optimal beamforming vector actually achieves the same maximum achievable rate as the optimal precoding matrix in multicast networks with two users. This also holds in the three-user case [7].

event will occur if $\log_2(1+P\gamma(\mathbf{x},\mathbf{H}))<\rho$, or equivalently, if $\gamma(\mathbf{x},\mathbf{H})<\frac{2^\rho-1}{P}$. Without loss of generality, we assume $\rho=1$ throughout this paper. Thus, $\frac{2^\rho-1}{P}=\frac{1}{P}$. Results for other values of ρ can be obtained similarly.

In the full-CSI case where all nodes in the multicast network know perfect CSI, the optimal beamforming vector is computed as[2]

$$\mathrm{Full}(\mathbf{H})=\arg\max_{\mathbf{x}\in\mathcal{X}}\gamma(\mathbf{x},\mathbf{H}).$$

Therefore, the full-CSI outage probability is

$$\mathtt{OUT}(\mathrm{Full})=\mathrm{Prob}\left\{\gamma(\mathrm{Full}(\mathbf{H}),\mathbf{H})<\frac{1}{P}\right\}=\mathrm{E}_{\mathbf{H}}\mathbf{1}_{\gamma(\mathrm{Full}(\mathbf{H}),\mathbf{H})<\frac{1}{P}}. \tag{1}$$

III. Channel Quantization and Encoding Rule

For an arbitrary quantizer Q, the distortion with respect to the outage probability is defined as

$$\mathsf{Dist}=\mathtt{OUT}(\mathsf{Q})-\mathtt{OUT}(\mathrm{Full}).$$

Since $\mathtt{OUT}(\mathrm{Full})$ is invariant for fixed P, minimizing Dist is equivalent to designing a quantizer to minimize $\mathtt{OUT}(\mathsf{Q})$. In the multicast network, we consider a global VLQ associated with a random codebook $\{\mathbf{x}_i\}_{\mathbb{N}}$ where $\mathbf{x}_i\in\mathcal{X}$ is independent and identically distributed with a uniform distribution on $\mathcal{X}$ for $i\in\mathbb{N}$ [8]. We omit the subscript $\mathbb{N}$ for notational convenience. The random codebook is generated each time the channel state changes and revealed to all nodes in the network. It provides a performance benchmark since if a random codebook can achieve certain performance, one deterministic codebook can be found to surpass this performance. For any realization of $\{\mathbf{x}_i\}$, the proposed VLQ is represented by

$$\mathsf{Q}_{\mathrm{VLQ}}=\{\mathbf{x}_i,\mathcal{R}_i,\mathtt{b}_i\}, \tag{2}$$

where $\mathcal{R}_i$ denotes the partition channel region of $\mathbf{x}_i$ for $i\in\mathbb{N}$ and $\mathbf{x}_i$ is used as the transmit beamforming vector when $\mathbf{H}\in\mathcal{R}_i$. Different from FLQs in which each partition channel region

[2]For any $\mathbf{H}$, $\mathrm{Full}(\mathbf{H})$ exists because $\gamma(\mathbf{x},\mathbf{H})$ is a continuous function on $\mathbf{x}$ and $\mathcal{X}$ is a bounded and closed set. There might exist more than one unit-normal vector that can achieve maximum value of $\gamma(\mathbf{x},\mathbf{H})$ and $\mathrm{Full}(\mathbf{H})$ can be any one of them.

consists of channel states that achieves the optimal performance with the "centroid" codeword, $\mathcal{R}_0$ in $\mathsf{Q}_{\mathrm{VLQ}}$ is set as

$$\mathcal{R}_0 = \left\{\mathbf{H} : \gamma\left(\mathbf{x}_0, \mathbf{H}\right) \geq \frac{1}{P}\right\} \bigcup \bigcap_{i \in \mathbb{N}} \left\{\mathbf{H} : \gamma\left(\mathbf{x}_i, \mathbf{H}\right) < \frac{1}{P}\right\}, \tag{3}$$

and $\mathcal{R}_i$ for $i \in \mathbb{N} - \{0\}$ is set as

$$\mathcal{R}_i = \left\{\mathbf{H} : \gamma\left(\mathbf{x}_i, \mathbf{H}\right) \geq \frac{1}{P}\right\} \bigcap \bigcap_{k=0}^{i-1} \left\{\mathbf{H} : \gamma\left(\mathbf{x}_k, \mathbf{H}\right) < \frac{1}{P}\right\}. \tag{4}$$

We can see that $\{\mathcal{R}_i\}$ is a collection of disjoint sets and the union of $\mathcal{R}_i$ for $i \in \mathbb{N}$ makes up the entire channel space. $\mathcal{R}_0$ is a union set of channel states for which using all vectors in the codebook cannot prevent outage and ones for which using $\mathbf{x}_0$ will not result in outage.[3] $\mathcal{R}_i$ for $i \in \mathbb{N} - \{0\}$ consists of channel states for which using $\mathbf{x}_i$ can prevent outage while using the preceding vectors $\mathbf{x}_0, \ldots, \mathbf{x}_{i-1}$ cannot.

Moreover, we let $\mathtt{b}_i$ denote the binary string that represents the index of $\mathbf{x}_i$. To be specific, we set $\mathtt{b}_0 = \epsilon$, which is an empty codeword[4], $\mathtt{b}_1 = \{0\}$, $\mathtt{b}_2 = \{1\}$, $\mathtt{b}_3 = \{00\}$, $\mathtt{b}_4 = \{01\}$ and so on for all codewords in the set $\{\epsilon, 0, 1, 00, 01, 10, 11, \ldots\}$. The length of $\mathtt{b}_i$ is $\lfloor \log_2(i+1) \rfloor$.

With perfect CSI and any realization of $\{\mathbf{x}_i\}$, $\mathsf{Q}_{\mathrm{VLQ}}$ first determines the partition channel region $\mathcal{R}_i$ in which the current channel state $\mathbf{H}$ falls according to (3) and (4). Then, the corresponding codeword $\mathbf{x}_i$ is chosen and $\lfloor \log_2(i+1) \rfloor$ bits are fed back to notify the index of $\mathbf{x}_i$. After decoding the feedback information, $\mathbf{x}_i$ is employed by the transmitter as the beamforming vector. Therefore, the average feedback rate of $\mathsf{Q}_{\mathrm{VLQ}}$ is

$$\mathtt{R}\left(\mathsf{Q}_{\mathrm{VLQ}}\right) = \sum_{i=1}^{\infty} \lfloor \log_2(i+1) \rfloor \mathrm{Prob}\left\{\mathbf{H} \in \mathcal{R}_i\right\} = \sum_{i=1}^{\infty} \lfloor \log_2(i+1) \rfloor \mathrm{E}_{\mathbf{H}} \mathrm{E}_{\{\mathbf{x}_i\}} \mathbf{1}_{\mathbf{H} \in \mathcal{R}_i}. \tag{5}$$

The outage probability is given by

$$\mathtt{OUT}(\mathsf{Q}_{\mathrm{VLQ}}) = \mathrm{E}_{\{\mathbf{x}_i\}} \mathrm{Prob}\left\{\gamma\left(\mathbf{x}_i, \mathbf{H}\right) < \frac{1}{P}, \forall i \in \mathbb{N}\right\} = \mathrm{E}_{\mathbf{H}} \mathrm{E}_{\{\mathbf{x}_i\}} \mathbf{1}_{\gamma(\mathbf{x}_i, \mathbf{H}) < \frac{1}{P}, \forall i \in \mathbb{N}}. \tag{6}$$

[3]It will be shown in Section IV that the channel region where using all codewords cannot prevent outage coincides with the region where even the optimal beamforming vector fails to avoid outage with probability one. Therefore, $\mathsf{Q}_{\mathrm{VLQ}}$ can determine whether $\mathbf{H}$ belongs to this region or not based on the expression of the optimal beamforming vector given by [6, Theorem 2], rather than checking all codewords in $\{\mathbf{x}_i\}$.

[4]An empty codeword is used here for illustration. Adding 1 bit to each codeword to avoid an empty codeword only increases the average feedback rate by 1 bit per channel realization, thereby not impacting the result of the average feedback rate being low.

IV. Outage Optimality

In this section, we show that the proposed VLQ in (2) will achieve the full-CSI outage probability.

Theorem 1. *For any $P > 0$, we have*

$$\boxed{\mathtt{OUT}(\mathsf{Q}_{\mathrm{VLQ}}) = \mathtt{OUT}(\mathrm{Full}).} \tag{7}$$

Proof: Define

$$\mathcal{S}_1 = \left\{\mathbf{H} : \mathbf{H} \in \mathbb{C}^{t\times 2}, \gamma\left(\mathrm{Full}\left(\mathbf{H}\right), \mathbf{H}\right) < \frac{1}{P}\right\}.$$

For any realization of $\{\mathbf{x}_i\}$, define

$$\mathcal{S}_2\left(\{\mathbf{x}_i\}\right) = \left\{\mathbf{H} : \mathbf{H} \in \mathbb{C}^{t\times 2}, \gamma\left(\mathbf{x}_i, \mathbf{H}\right) < \frac{1}{P}, \forall i \in \mathbb{N}\right\}.$$

For brevity, we omit the dependency of $\mathcal{S}_2\left(\{\mathbf{x}_i\}\right)$ on $\{\mathbf{x}_i\}$ and simply use $\mathcal{S}_2$. From (1) and (6), $\mathtt{OUT}\left(\mathrm{Full}\right)$ and $\mathtt{OUT}(\mathsf{Q}_{\mathrm{VLQ}})$ can be rewritten as

$$\begin{aligned} \mathtt{OUT}\left(\mathrm{Full}\right) &= \mathrm{E}_{\mathbf{H}} \mathbf{1}_{\mathbf{H}\in\mathcal{S}_1}, & (8)\\ \mathtt{OUT}(\mathsf{Q}_{\mathrm{VLQ}}) &= \mathrm{E}_{\{\mathbf{x}_i\}} \mathrm{E}_{\mathbf{H}} \mathbf{1}_{\mathbf{H}\in\mathcal{S}_2}. & (9) \end{aligned}$$

For convenience, we define

$$\begin{aligned} \mathcal{S}_{21} &= \left\{\mathbf{H} : \mathbf{H} \in \mathcal{S}_2, \gamma\left(\mathrm{Full}\left(\mathbf{H}\right), \mathbf{H}\right) < \tfrac{1}{P}\right\},\\ \mathcal{S}_{22} &= \left\{\mathbf{H} : \mathbf{H} \in \mathcal{S}_2, \gamma\left(\mathrm{Full}\left(\mathbf{H}\right), \mathbf{H}\right) = \tfrac{1}{P}\right\},\\ \mathcal{S}_{23} &= \left\{\mathbf{H} : \mathbf{H} \in \mathcal{S}_2, \gamma\left(\mathrm{Full}\left(\mathbf{H}\right), \mathbf{H}\right) > \tfrac{1}{P}\right\}. \end{aligned}$$

Since $\mathcal{S}_2 = \mathcal{S}_{21} \cup \mathcal{S}_{22} \cup \mathcal{S}_{23}$ and $\mathcal{S}_{21}$, $\mathcal{S}_{22}$, $\mathcal{S}_{23}$ are mutually exclusive, $\mathtt{OUT}\left(\mathsf{Q}_{\mathrm{VLQ}}\right)$ in (9) is rewritten as

$$\mathtt{OUT}\left(\mathsf{Q}_{\mathrm{VLQ}}\right) = \sum_{l=1}^{3} \mathrm{E}_{\{\mathbf{x}_i\}} \mathrm{E}_{\mathbf{H}} \mathbf{1}_{\mathbf{H}\in\mathcal{S}_{2l}}. \tag{10}$$

In order to prove $\mathtt{OUT}(\mathsf{Q}_{\mathrm{VLQ}}) = \mathtt{OUT}(\mathrm{Full})$, from (8) and (10), we will show $\mathrm{E}_{\{\mathbf{x}_i\}} \mathrm{E}_{\mathbf{H}} \mathbf{1}_{\mathbf{H}\in\mathcal{S}_{21}} = \mathrm{E}_{\mathbf{H}} \mathbf{1}_{\mathbf{H}\in\mathcal{S}_1}$, $\mathrm{E}_{\{\mathbf{x}_i\}} \mathrm{E}_{\mathbf{H}} \mathbf{1}_{\mathbf{H}\in\mathcal{S}_{22}} = 0$ and $\mathrm{E}_{\{\mathbf{x}_i\}} \mathrm{E}_{\mathbf{H}} \mathbf{1}_{\mathbf{H}\in\mathcal{S}_{23}} = 0$.

First, to prove $\mathrm{E}_{\{\mathbf{x}_i\}} \mathrm{E}_{\mathbf{H}} \mathbf{1}_{\mathbf{H}\in\mathcal{S}_{21}} = \mathrm{E}_{\mathbf{H}} \mathbf{1}_{\mathbf{H}\in\mathcal{S}_1}$, it is sufficient to prove $\mathbf{1}_{\mathbf{H}\in\mathcal{S}_1} = \mathbf{1}_{\mathbf{H}\in\mathcal{S}_{21}}$ for any realizations of $\mathbf{H}$ and $\{\mathbf{x}_i\}$. When $\mathbf{1}_{\mathbf{H}\in\mathcal{S}_1} = 0$, it means $\mathbf{H} \notin \mathcal{S}_1$ and $\gamma\left(\mathrm{Full}\left(\mathbf{H}\right), \mathbf{H}\right) \geq \frac{1}{P}$. Then, $\mathbf{H} \notin \mathcal{S}_{21}$ and $\mathbf{1}_{\mathbf{H}\in\mathcal{S}_{21}} = 0$. When $\mathbf{1}_{\mathbf{H}\in\mathcal{S}_1} = 1$, $\mathbf{H} \in \mathcal{S}_1$. By the optimality of $\mathrm{Full}\left(\mathbf{H}\right)$, $\mathbf{H} \in \mathcal{S}_2$.

Since $\mathcal{S}_{21} = \mathcal{S}_1 \cap \mathcal{S}_2$, $\mathbf{H} \in \mathcal{S}_{21}$ and $\mathbf{1}_{\mathbf{H}\in\mathcal{S}_{21}} = 1$. Since $\mathbf{1}_{\mathbf{H}\in\mathcal{S}_{21}}$ and $\mathbf{1}_{\mathbf{H}\in\mathcal{S}_1}$ only take values at 0 and 1, $\mathbf{1}_{\mathbf{H}\in\mathcal{S}_{21}} = \mathbf{1}_{\mathbf{H}\in\mathcal{S}_1}$. Therefore, $\mathrm{E}_{\{\mathbf{x}_i\}}\mathrm{E}_{\mathbf{H}}\mathbf{1}_{\mathbf{H}\in\mathcal{S}_{21}} = \mathrm{E}_{\mathbf{H}}\mathbf{1}_{\mathbf{H}\in\mathcal{S}_1}$.

Second, we will prove $\mathrm{E}_{\{\mathbf{x}_i\}}\mathrm{E}_{\mathbf{H}}\mathbf{1}_{\mathbf{H}\in\mathcal{S}_{22}} = 0$. Define

$$\mathcal{S}_3 = \left\{\mathbf{H} : \mathbf{H} \in \mathbb{C}^{t\times 2}, \gamma\left(\mathrm{Full}\left(\mathbf{H}\right), \mathbf{H}\right) = \frac{1}{P}\right\}.$$

By definition, $\mathcal{S}_{22} = \mathcal{S}_2 \cap \mathcal{S}_3$. Then, $\mathrm{E}_{\{\mathbf{x}_i\}}\mathrm{E}_{\mathbf{H}}\mathbf{1}_{\mathbf{H}\in\mathcal{S}_{22}} \leq \mathrm{E}_{\mathbf{H}}\mathbf{1}_{\mathbf{H}\in\mathcal{S}_3} = \mathrm{Prob}\left\{\gamma\left(\mathrm{Full}\left(\mathbf{H}\right), \mathbf{H}\right) = \frac{1}{P}\right\}$. Define $m_{\min} = \arg\min_{m=1,2} \chi_m$, $m_{\max} = \arg\max_{m=1,2} \chi_m$ and $\theta = \frac{\left|\mathbf{h}_1^\dagger \mathbf{h}_2\right|^2}{\chi_1\chi_2}$. According to [6, Theorem 2],

$$\gamma\left(\mathrm{Full}\left(\mathbf{H}\right), \mathbf{H}\right) = \begin{cases} \chi_{m_{\min}}, & \theta \geq \frac{\chi_{m_{\min}}}{\chi_{m_{\max}}}, \\ \frac{\chi_{m_{\min}}}{1+\beta^2}, & \theta < \frac{\chi_{m_{\min}}}{\chi_{m_{\max}}}, \end{cases} \tag{11}$$

where $\beta = \frac{\sqrt{\chi_{m_{\min}}} - \sqrt{\chi_{m_{\max}}\theta}}{\sqrt{\chi_{m_{\max}} - \chi_{m_{\max}}\theta}}$. Since θ, χ_1, and χ_2 are mutually independent, θ and $\chi_{m_{\min}}$ and $\chi_{m_{\max}}$ are also mutually independent [9]. Therefore, we obtain

$$\begin{aligned} &\mathrm{Prob}\left\{\gamma\left(\mathrm{Full}\left(\mathbf{H}\right), \mathbf{H}\right) = \frac{1}{P}\right\} \\ &= \mathrm{Prob}\left(\theta \geq \frac{\chi_{m_{\min}}}{\chi_{m_{\max}}}, \chi_{m_{\min}} = \frac{1}{P}\right) + \mathrm{Prob}\left(\theta < \frac{\chi_{m_{\min}}}{\chi_{m_{\max}}}, \frac{\chi_{m_{\min}}}{1+\beta^2} = \frac{1}{P}\right) \\ &\leq \mathrm{Prob}\left(\chi_{m_{\min}} = \frac{1}{P}\right) + \mathrm{Prob}\left(\frac{\chi_{m_{\min}}}{1+\beta^2} = \frac{1}{P}\right). \end{aligned}$$

Since the probability that a continuous r.v. assumes a specific value is zero, $\mathrm{Prob}\left(\chi_{m_{\min}} = \frac{1}{P}\right) = 0$. Moreover, $\mathrm{Prob}\left(\frac{\chi_{m_{\min}}}{1+\beta^2} = \frac{1}{P}\right)$ can be rewritten as

$$\mathrm{Prob}\left(\frac{\chi_{m_{\min}}}{1+\beta^2} = \frac{1}{P}\right) = \mathrm{E}_{\chi_{m_{\min}},\chi_{m_{\max}}}\mathrm{Prob}\left\{\theta = g_1 \text{ or } \theta = g_2\right\},$$

where [5]

$$g_1 = \left[\frac{1}{P\sqrt{\chi_{m_{\min}}\chi_{m_{\max}}}} + \sqrt{\left(1 - \frac{1}{P\chi_{m_{\min}}}\right)\left(1 - \frac{1}{P\chi_{m_{\max}}}\right)}\right]^2,$$

$$g_2 = \left[\frac{1}{P\sqrt{\chi_{m_{\min}}\chi_{m_{\max}}}} - \sqrt{\left(1 - \frac{1}{P\chi_{m_{\min}}}\right)\left(1 - \frac{1}{P\chi_{m_{\max}}}\right)}\right]^2.$$

For fixed $\chi_{m_{\min}}$ and $\chi_{m_{\max}}$, we obtain

$$\mathrm{Prob}\left\{\theta = g_1 \text{ or } \theta = g_2\right\} \leq \mathrm{Prob}\left\{\theta = g_1\right\} + \mathrm{Prob}\left\{\theta = g_2\right\}.$$

[5] g_1 and g_2 are obtained by substituting $\beta = \frac{\sqrt{\chi_{m_{\min}}} - \sqrt{\chi_{m_{\max}}\theta}}{\sqrt{\chi_{m_{\max}} - \chi_{m_{\max}}\theta}}$ into $\frac{\chi_{m_{\min}}}{1+\beta^2} = \frac{1}{P}$ and solving it as a quadratic equation with respect to $\sqrt{\theta}$.

Since Prob $\{\theta = g_l\} = 0$ for $l = 1, 2$, Prob $\{\theta = g_1 \text{ or } \theta = g_2\} = 0$. It follows that Prob $\left(\frac{\chi m_{\min}}{1+\beta^2} = \frac{1}{P}\right) = 0$, Prob $\left\{\gamma\left(\text{Full}\left(\mathbf{H}\right), \mathbf{H}\right) = \frac{1}{P}\right\} \leq 0$ and $\mathrm{E}_{\{\mathbf{x}_i\}}\mathrm{E}_{\mathbf{H}}\mathbf{1}_{\mathbf{H}\in\mathcal{S}_{22}} \leq 0$. Since the probability is non-negative, $\mathrm{E}_{\{\mathbf{x}_i\}}\mathrm{E}_{\mathbf{H}}\mathbf{1}_{\mathbf{H}\in\mathcal{S}_{22}} = 0$.

Finally, we will prove $\mathrm{E}_{\{\mathbf{x}_i\}}\mathrm{E}_{\mathbf{H}}\mathbf{1}_{\mathbf{H}\in\mathcal{S}_{23}} = 0$. Define

$$\mathcal{S}_4 = \left\{\mathbf{H} : \mathbf{H} \in \mathbb{C}^{t\times 2}, \gamma\left(\text{Full}\left(\mathbf{H}\right), \mathbf{H}\right) > \frac{1}{P}\right\}.$$

Since $\mathcal{S}_{23} = \mathcal{S}_2 \cap \mathcal{S}_4$, it can be seen that

$$\mathrm{E}_{\{\mathbf{x}_i\}}\mathrm{E}_{\mathbf{H}}\mathbf{1}_{\mathbf{H}\in\mathcal{S}_{23}} = \int_{\mathbf{H}\in\mathcal{S}_4} f_{\mathbf{H}}(\mathbf{H})\mathrm{E}_{\{\mathbf{x_i}\}}\mathbf{1}_{\mathbf{H}\in\mathcal{S}_2}\mathrm{d}\mathbf{H}.$$

Then, to prove $\mathrm{E}_{\{\mathbf{x}_i\}}\mathrm{E}_{\mathbf{H}}\mathbf{1}_{\mathbf{H}\in\mathcal{S}_{23}} = 0$, it is sufficicent to show $\mathrm{E}_{\{\mathbf{x}_i\}}\mathbf{1}_{\mathbf{H}\in\mathcal{S}_2} = 0$ for any $\mathbf{H} \in \mathcal{S}_4$. By contradiction, assume $\exists\tilde{\mathbf{H}} \in \mathcal{S}_4$, s.t. $\mathrm{E}_{\{\mathbf{x}_i\}}\mathbf{1}_{\tilde{\mathbf{H}}\in\mathcal{S}_2} = \varepsilon > 0$. In contrast,

$$\begin{aligned}
\mathrm{E}_{\{\mathbf{x}_i\}}\mathbf{1}_{\tilde{\mathbf{H}}\in\mathcal{S}_2} &= \text{Prob}\left\{\gamma\left(\mathbf{x}_i, \tilde{\mathbf{H}}\right) < \frac{1}{P}, \forall i \in \mathbb{N}\right\} \\
&\leq \text{Prob}\left\{\gamma\left(\mathbf{x}_i, \tilde{\mathbf{H}}\right) < \frac{1}{P}, \forall i \in \{0, 1, \ldots, K-1\}\right\} \\
&= \left[\text{Prob}\left\{\gamma\left(\mathbf{x}_i, \tilde{\mathbf{H}}\right) < \frac{1}{P}\right\}\right]^K, \qquad (12)
\end{aligned}$$

where $K \geq 1$ can be any finite natural number. We shall use the following lemma, the proof of which is provided in Appendix A.

Lemma 1. *If* $\gamma\left(\text{Full}\left(\mathbf{H}\right), \mathbf{H}\right) > \frac{1}{P}$*, there exists* $\Pi \in (0, 1)$ *such that for any* $\mathbf{x} \in \mathcal{X}$ *with* $\left|\mathbf{x}^\dagger\text{Full}\left(\mathbf{H}\right)\right|^2 \geq \Pi$, $\gamma\left(\mathbf{x}, \mathbf{H}\right) \geq \frac{1}{P}$ *holds.*

From lemma 1, for a given $\tilde{\mathbf{H}}$, we have

$$\text{Prob}\left\{\gamma\left(\mathbf{x}_i, \tilde{\mathbf{H}}\right) \geq \frac{1}{P}\right\} \geq \text{Prob}\left\{\left|\mathbf{x}_i^\dagger\text{Full}\left(\tilde{\mathbf{H}}\right)\right|^2 \geq \Pi\right\} = (1-\Pi)^{t-1} > 0.$$

Therefore, Prob $\left\{\gamma\left(\mathbf{x}_i, \tilde{\mathbf{H}}\right) < \frac{1}{P}\right\} \leq 1 - (1-\Pi)^{t-1} < 1$. By (12), it can be derived that $\mathrm{E}_{\{\mathbf{x}_i\}}\mathbf{1}_{\tilde{\mathbf{H}}\in\mathcal{S}_2} \leq \left[1 - (1-\Pi)^{t-1}\right]^K$. Since K can be chosen to be arbitrarily large, $\mathrm{E}_{\{\mathbf{x}_i\}}\mathbf{1}_{\tilde{\mathbf{H}}\in\mathcal{S}_2} \leq 0$ must hold. Then, $\mathrm{E}_{\{\mathbf{x}_i\}}\mathbf{1}_{\tilde{\mathbf{H}}\in\mathcal{S}_2} = 0$ and $\mathrm{E}_{\{\mathbf{x}_i\}}\mathrm{E}_{\mathbf{H}}\mathbf{1}_{\mathbf{H}\in\mathcal{S}_{23}} = 0$, which completes the proof. ■

Remark 1: From the proof above, it can be seen that for any given $\mathbf{H}$, if the optimal beamforming vector $\text{Full}\left(\mathbf{H}\right)$ is able to make the channel strictly non-outage (i.e., $\gamma\left(\text{Full}\left(\mathbf{H}\right), \mathbf{H}\right) > \frac{1}{P}$), there must exist a certain region in the unit sphere for beamforming vectors with non-zero probability where all the unit-normal vectors will also result in non-outage. On the other hand, the infinite vectors in the random codebook ensure that at least one efficient vector in that region will eventually be chosen, thus making the channel state non-outage. This also explains why a FLQ with a finite-cardinality codebook cannot achieve the full-CSI outage probability.

V. Average Feedback Rate

In this section, we present an upper bound on the average feedback rate of Q_{VLQ} based on the random codebook $\{\mathbf{x}_i\}$.

Theorem 2. *For any $P > 0$, we have*

$$\boxed{\mathtt{R}\left(\mathsf{Q}_{\text{VLQ}}\right) \leq C_0 e^{-\frac{1}{P}} \left[\frac{1}{P} + \frac{1}{P^{2t}} + \frac{\log\left(1+P\right)}{P}\right],} \tag{13}$$

where $C_0 > 0$ is a constant that is independent of P.

Proof: Define

$$\begin{aligned}
\mathcal{H}_0 &= \left\{\mathbf{H} : \mathbf{H} \in \mathtt{C}^{t\times 2}, \chi_1 \geq \tfrac{1}{P}, \chi_2 \geq \tfrac{1}{P}\right\}, \\
\mathcal{H}_1 &= \left\{\mathbf{H} : \mathbf{H} \in \mathcal{H}_0, \gamma\left(\text{Full}\left(\mathbf{H}\right), \mathbf{H}\right) < \tfrac{1}{P}\right\}, \\
\mathcal{H}_2 &= \left\{\mathbf{H} : \mathbf{H} \in \mathcal{H}_0, \gamma\left(\text{Full}\left(\mathbf{H}\right), \mathbf{H}\right) = \tfrac{1}{P}\right\}, \\
\mathcal{H}_3 &= \left\{\mathbf{H} : \mathbf{H} \in \mathcal{H}_0, \gamma\left(\text{Full}\left(\mathbf{H}\right), \mathbf{H}\right) > \tfrac{1}{P}\right\}.
\end{aligned}$$

Based on the encoding rule of Q_{VLQ} and the random codebook $\{\mathbf{x}_i\}$, the feedback rate in (5) can be rewritten as

$$\mathtt{R}\left(\mathsf{Q}_{\text{VLQ}}\right) = \sum_{l=1}^{3} \int_{\mathbf{H}\in\mathcal{H}_l} \Phi f_{\mathbf{H}}(\mathbf{H}) \mathrm{d}\mathbf{H},$$

where

$$\Phi = \sum_{i=1}^{\infty} p^i (1-p) \lfloor \log_2(i+1) \rfloor, p = \text{Prob}\left\{\gamma(\mathbf{x}_i, \mathbf{H}) < \frac{1}{P}\right\}.$$

For any $\mathbf{H} \in \mathcal{H}_1$, $p = 1$ and $\Phi = 0$; for any $\mathbf{H} \in \mathcal{H}_2$, from the proof in Theorem 1, $p = 1$ and $\Phi = 0$. Then, $\int_{\mathbf{H}\in\mathcal{H}_1} \Phi f_{\mathbf{H}}(\mathbf{H})\mathrm{d}\mathbf{H} = \int_{\mathbf{H}\in\mathcal{H}_2} \Phi f_{\mathbf{H}}(\mathbf{H})\mathrm{d}\mathbf{H} = 0$, and $\mathtt{R}\left(\mathsf{Q}_{\text{VLQ}}\right)$ is equivalent to

$$\mathtt{R}\left(\mathsf{Q}_{\text{VLQ}}\right) = \int_{\mathbf{H}\in\mathcal{H}_3} \Phi f_{\mathbf{H}}(\mathbf{H}) \mathrm{d}\mathbf{H}. \tag{14}$$

The following lemma exhibits an upper bound on Φ, the proof of which is presented in Appendix B.

Lemma 2. *For any* $0 \leq p < 1$, *we have*

$$\Phi \leq p(1-p) + \left(\frac{6}{\log 2} + 2\right) p^2 + \frac{2}{\log 2} p^2 \log \frac{1}{1-p}. \tag{15}$$

Applying (15) to (14), it follows that

$$\mathtt{R}\left(\mathsf{Q}_{\text{VLQ}}\right) \leq I_1 + I_2 + I_3, \tag{16}$$

where

$$I_1 = C_1 \int_{\mathbf{H} \in \mathcal{H}_3} p(1-p) f_{\mathbf{H}}(\mathbf{H}) \mathrm{d}\mathbf{H},$$
$$I_2 = C_2 \int_{\mathbf{H} \in \mathcal{H}_3} p^2 f_{\mathbf{H}}(\mathbf{H}) \mathrm{d}\mathbf{H},$$
$$I_3 = C_3 \int_{\mathbf{H} \in \mathcal{H}_3} p^2 \left(\log \frac{1}{1-p}\right) f_{\mathbf{H}}(\mathbf{H}) \mathrm{d}\mathbf{H},$$

and $C_1 = 1$, $C_2 = \frac{6}{\log 2} + 2$, $C_3 = \frac{2}{\log 2}$.

To proceed, we first present useful upper and lower bounds on p. For an upper bound on p, using [10, Lemma 2] and [9], we derive that

$$p \leq \sum_{m=1}^{2} \text{Prob}\left\{\left|\mathbf{x}_i^\dagger \mathbf{h}_m\right|^2 < \frac{1}{P}\right\} = \sum_{m=1}^{2} \left[1 - \left(1 - \frac{1}{P\chi_m}\right)^{t-1}\right],$$

where the last equality arises from the fact that $\text{Prob}\left\{\left|\mathbf{x}_i^\dagger \mathbf{h}_m\right|^2 < x\right\} = 1 - \left(1 - \frac{x}{\chi_m}\right)^{t-1}$ [9]. Since $(1-a)^b \geq 1 - ab$ for $0 < a < 1$ and $b \geq 1$, $\left(1 - \frac{1}{P\chi_m}\right)^{t-1} \geq 1 - \frac{t-1}{P\chi_m}$. Therefore, p is upper-bounded by

$$p \leq \frac{t-1}{P} \sum_{m=1}^{2} \frac{1}{\chi_m}. \tag{17}$$

Another upper bound on p obtained from Lemma 1 and its proof in Appendix A is given as

$$p \leq 1 - (1 - \Pi)^{t-1}, \tag{18}$$

where

$$\Pi = 1 - \min_{m=1,2} \left[\frac{\left|\left[\text{Full}\left(\mathbf{H}\right)\right]^\dagger \mathbf{h}_m\right|^2 - \frac{1}{P}}{\chi_m}\right]^2.$$

In addition, a lower bound on p (or equivalently, the upper bound on $1-p$) is obtained as

$$1 - p \leq \text{Prob}\left\{\left|\mathbf{x}_i^\dagger \mathbf{h}_1\right|^2 \geq \frac{1}{P}\right\} = \left(1 - \frac{1}{P\chi_1}\right)^{t-1}. \tag{19}$$

To derive an upper bound on I_1, since $\mathcal{H}_3 \subseteq \mathcal{H}_0$, we get

$$I_1 \leq C_1 \int_{\mathbf{H}\in\mathcal{H}_0} p(1-p) f_{\mathbf{H}}(\mathbf{H}) \mathrm{d}\mathbf{H}.$$

Substituting the upper bounds in (17) and (19) into I_1, it can be deduced that

$$I_1 \leq \frac{C_4}{P} \sum_{m=1}^{2} \int_{\mathbf{H}\in\mathcal{H}_0} \frac{1}{\chi_m} \left(1 - \frac{1}{P\chi_1}\right)^{t-1} f_{\mathbf{H}}(\mathbf{H}) \mathrm{d}\mathbf{H},$$

where $C_4 = (t-1)C_1$. Since χ_m is chi-squared distributed, the PDF of χ_m is $f_{\chi_m}(\chi_m) = \frac{\chi_m^{t-1} e^{-\chi_m}}{(t-1)!}$ for $m = 1, 2$ [11]. Then, we obtain

$$\begin{aligned}
I_1 &\leq \frac{C_4}{P} \sum_{m=1}^{2} \int_{\frac{1}{P}}^{\infty} \int_{\frac{1}{P}}^{\infty} \frac{1}{\chi_m} \left(1 - \frac{1}{P\chi_1}\right)^{t-1} \left[\frac{\chi_1^{t-1} e^{-\chi_1}}{(t-1)!}\right] \left[\frac{\chi_2^{t-1} e^{-\chi_2}}{(t-1)!}\right] \mathrm{d}\chi_1 \mathrm{d}\chi_2 \\
&= \frac{C_5}{P} \int_{\frac{1}{P}}^{\infty} \left(1 - \frac{1}{P\chi_1}\right)^{t-1} \chi_1^{t-2} e^{-\chi_1} \mathrm{d}\chi_1 \int_{\frac{1}{P}}^{\infty} \frac{\chi_2^{t-1} e^{-\chi_2}}{(t-1)!} \mathrm{d}\chi_2 \\
&+ \frac{C_5}{P} \int_{\frac{1}{P}}^{\infty} \left(1 - \frac{1}{P\chi_1}\right)^{t-1} \chi_1^{t-1} e^{-\chi_1} \mathrm{d}\chi_1 \int_{\frac{1}{P}}^{\infty} \frac{\chi_2^{t-2} e^{-\chi_2}}{(t-1)!} \mathrm{d}\chi_2,
\end{aligned}$$

where $C_5 = \frac{C_4}{(t-1)!}$. Noting that $\int_0^\infty x^{n-1} e^{-x} \mathrm{d}x = (n-1)!$ for $n \geq 1$ and $n \in \mathbb{N}$ [11], I_1 is bounded by

$$\begin{aligned}
I_1 &\leq \frac{C_5}{P} \int_{\frac{1}{P}}^{\infty} \left(1 - \frac{1}{P\chi_1}\right)^{t-1} \chi_1^{t-2} e^{-\chi_1} \mathrm{d}\chi_1 \int_{0}^{\infty} \frac{\chi_2^{t-1} e^{-\chi_2}}{(t-1)!} \mathrm{d}\chi_2 \\
&+ \frac{C_5}{P} \int_{\frac{1}{P}}^{\infty} \left(1 - \frac{1}{P\chi_1}\right)^{t-1} \chi_1^{t-1} e^{-\chi_1} \mathrm{d}\chi_1 \int_{0}^{\infty} \frac{\chi_2^{t-2} e^{-\chi_2}}{(t-1)!} \mathrm{d}\chi_2 \\
&\leq \frac{C_5}{P} \int_{\frac{1}{P}}^{\infty} \left(1 - \frac{1}{P\chi_1}\right)^{t-1} \chi_1^{t-2} e^{-\chi_1} \mathrm{d}\chi_1 + \frac{C_6}{P} \int_{\frac{1}{P}}^{\infty} \left(1 - \frac{1}{P\chi_1}\right)^{t-1} \chi_1^{t-1} e^{-\chi_1} \mathrm{d}\chi_1,
\end{aligned}$$

where $C_6 = \frac{C_5}{t-1}$. Letting $\chi_1 - \frac{1}{P} = \lambda_1$, the bound is derived as

$$\begin{aligned}
I_1 &\leq C_5 \frac{e^{-\frac{1}{P}}}{P} \int_0^\infty \frac{\lambda_1}{\lambda_1 + \frac{1}{P}} \lambda_1^{t-2} e^{-\lambda_1} \mathrm{d}\lambda_1 + C_6 \frac{e^{-\frac{1}{P}}}{P} \int_0^\infty \lambda_1^{t-1} e^{-\lambda_1} \mathrm{d}\lambda_1 \\
&\leq C_5 \frac{e^{-\frac{1}{P}}}{P} \int_0^\infty \lambda_1^{t-2} e^{-\lambda_1} \mathrm{d}\lambda_1 + C_6 (t-1)! \frac{e^{-\frac{1}{P}}}{P} \\
&= C_5 (t-2)! \frac{e^{-\frac{1}{P}}}{P} + C_6 (t-1)! \frac{e^{-\frac{1}{P}}}{P} = C_7 \frac{e^{-\frac{1}{P}}}{P}, \qquad (20)
\end{aligned}$$

where $C_7 = (t-2)! C_5 + (t-1)! C_6$.

To derive I_2, applying the upper bound in (17) and based on the fact that $\mathcal{H}_1 \subseteq \mathcal{H}_0$, we obtain

$$
\begin{aligned}
I_2 &\leq \frac{C_8}{P^2}\int_{\mathbf{H}\in\mathcal{H}_0}\left[\frac{1}{\chi_1}+\frac{1}{\chi_2}\right]^2 f_{\mathbf{H}}(\mathbf{H})\mathrm{d}\mathbf{H} \\
&= \frac{C_8}{P^2}\int_{\frac{1}{P}}^{\infty}\int_{\frac{1}{P}}^{\infty}\left[\frac{1}{\chi_1^2}+\frac{1}{\chi_2^2}+\frac{2}{\chi_1\chi_2}\right]\left[\frac{\chi_1^{t-1}e^{-\chi_1}}{(t-1)!}\right]\left[\frac{\chi_2^{t-1}e^{-\chi_2}}{(t-1)!}\right]\mathrm{d}\chi_1\mathrm{d}\chi_2 \\
&= \frac{2C_8}{(t-1)!P^2}\int_{\frac{1}{P}}^{\infty}\chi_1^{t-3}e^{-\chi_1}\mathrm{d}\chi_1\int_{\frac{1}{P}}^{\infty}\frac{\chi_2^{t-1}e^{-\chi_2}}{(t-1)!}\mathrm{d}\chi_2+\frac{2C_8}{(t-1)!P^2}\int_{\frac{1}{P}}^{\infty}\chi_1^{t-2}e^{-\chi_1}\mathrm{d}\chi_1\int_{\frac{1}{P}}^{\infty}\frac{\chi_2^{t-2}e^{-\chi_2}}{(t-1)!}\mathrm{d}\chi_2 \\
&\leq \frac{C_9}{P^2}\int_{\frac{1}{P}}^{\infty}\chi_1^{t-3}e^{-\chi_1}\mathrm{d}\chi_1\int_{0}^{\infty}\frac{e^{-\chi_2}\chi_2^{t-1}}{(t-1)!}\mathrm{d}\chi_2+\frac{C_9}{P^2}\int_{\frac{1}{P}}^{\infty}\chi_1^{t-2}e^{-\chi_1}\mathrm{d}\chi_1\int_{0}^{\infty}\frac{\chi_2^{t-2}e^{-\chi_2}}{(t-1)!}\mathrm{d}\chi_2 \\
&= \frac{C_9}{P^2}\int_{\frac{1}{P}}^{\infty}\chi_1^{t-3}e^{-\chi_1}\mathrm{d}\chi_1+\frac{C_{10}}{P^2}\int_{\frac{1}{P}}^{\infty}\chi_1^{t-2}e^{-\chi_1}\mathrm{d}\chi_1,
\end{aligned}
$$

where $C_8 = (t-1)^2 C_2$, $C_9 = \frac{2C_8}{(t-1)!}$ and $C_{10} = \frac{C_9}{t-1}$. When $t \geq 3$, I_2 is upper-bounded by

$$
I_2 \leq \frac{C_9}{P^2}\Gamma\left(t-2,\frac{1}{P}\right)+\frac{C_{10}}{P^2}\Gamma\left(t-1,\frac{1}{P}\right), \tag{21}
$$

where $\Gamma(n,a) = \int_a^{\infty} x^{n-1}e^{-x}\mathrm{d}x$ for $n > 0, a > 0$, is the incomplete gamma function. The following lemma shows an upper bound on the incomplete gamma function, the proof of which is in Appendix C.

Lemma 3. *For* $n > 0, n \in \mathbb{N}$ *and* $a > 0$, *we have*

$$
\Gamma(n,a) \leq n!e^{-a}\left(1+\alpha^{n-1}\right). \tag{22}
$$

Applying (22) to (21) yields

$$
\begin{aligned}
I_2 &\leq \frac{C_9}{P^2}(t-2)!e^{-\frac{1}{P}}\left(1+\frac{1}{P^{t-3}}\right)+\frac{C_{10}}{P^2}(t-1)!e^{-\frac{1}{P}}\left(1+\frac{1}{P^{t-2}}\right) \\
&= C_{11}\frac{e^{-\frac{1}{P}}}{P^2}+C_{12}\frac{e^{-\frac{1}{P}}}{P^{t-1}}+C_{13}\frac{e^{-\frac{1}{P}}}{P^t},
\end{aligned} \tag{23}
$$

where $C_{11} = C_9(t-2)!+C_{10}(t-1)!$, $C_{12} = C_9(t-2)!$ and $C_{13} = C_{10}(t-1)!$. When $t = 2$, the upper bound on I_2 is

$$
\begin{aligned}
I_2 &\leq \frac{C_9}{P^2}\int_{\frac{1}{P}}^{\infty}\frac{e^{-\chi_1}}{\chi_1}\mathrm{d}\chi_1+\frac{C_{10}}{P^2}\int_{\frac{1}{P}}^{\infty}e^{-\chi_1}\mathrm{d}\chi_1 \\
&= \frac{C_9}{P^2}\mathtt{E}_1\left(\frac{1}{P}\right)+C_{10}\frac{e^{-\frac{1}{P}}}{P^2} \\
&\leq C_9\frac{e^{-\frac{1}{P}}}{P^2}\log(1+P)+C_{10}\frac{e^{-\frac{1}{P}}}{P^2},
\end{aligned} \tag{24}
$$

where $\mathtt{E}_1(z)=\int_z^{\infty}\frac{e^{-z}}{z}\mathrm{d}z$ is the exponential integral with an upper bound as $\mathtt{E}_1(z)\le e^{-z}\log\left(1+\frac{1}{z}\right)$ [11]. From (23) and (24), the upper bound on I_2 for any $t\ge 2$ can be obtained as

$$\begin{aligned}I_2 &\le \left[C_{11}\frac{e^{-\frac{1}{P}}}{P^2}+C_{12}\frac{e^{-\frac{1}{P}}}{P^{t-1}}+C_{13}\frac{e^{-\frac{1}{P}}}{P^t}\right]\times \mathbf{1}_{t\ge 3}+\left[C_9\frac{e^{-\frac{1}{P}}}{P^2}\log(1+P)+C_{10}\frac{e^{-\frac{1}{P}}}{P^2}\right]\times\mathbf{1}_{t=2}\\ &\le C_{14}e^{-\frac{1}{P}}\left[\frac{1}{P}+\frac{1}{P^{2t}}+\frac{\log(1+P)}{P}\right],\end{aligned}\tag{25}$$

where $C_{14}=[C_{11}+C_{12}+C_{13}]\times\mathbf{1}_{t\ge 3}+[C_9+C_{10}]\times\mathbf{1}_{t=2}$. The last inequality is because when $0<P\le 1$, $\frac{1}{P^2}\le\frac{1}{P^{2t}}$, $\frac{1}{P^{t-1}}\le\frac{1}{P^{2t}}$, $\frac{1}{P^t}\le\frac{1}{P^{2t}}$, $\frac{\log(1+P)}{P^2}\le\frac{1}{P^2}\le\frac{1}{P^{2t}}$; when $P>1$, $\frac{1}{P^2}\le\frac{1}{P}$, $\frac{1}{P^{t-1}}\le\frac{1}{P}$, $\frac{1}{P^t}\le\frac{1}{P}$, $\frac{\log(1+P)}{P^2}\le\frac{\log(1+P)}{P}$.

To derive I_3, we need to find an upper bound on $\log\frac{1}{1-p}$ first. By applying (18), we obtain

$$\log\frac{1}{1-p}\le 2(t-1)\log\frac{1}{\min\limits_{m=1,2}\frac{\left|[\mathrm{Full}(\mathbf{H})]^{\dagger}\mathbf{h}_m\right|^2-\frac{1}{P}}{\chi_m}}.$$

From (11), it can be found that when $\theta\ge\frac{\chi_{m_{\min}}}{\chi_{m_{\max}}}$, $\min\limits_{m=1,2}\frac{\left|[\mathrm{Full}(\mathbf{H})]^{\dagger}\mathbf{h}_m\right|^2-\frac{1}{P}}{\chi_m}\ge\frac{\chi_{m_{\min}}-\frac{1}{P}}{\chi_{m_{\max}}}=\frac{\gamma(\mathrm{Full}(\mathbf{H}),\mathbf{H})-\frac{1}{P}}{\chi_{m_{\max}}}$; when $\theta<\frac{\chi_{m_{\min}}}{\chi_{m_{\max}}}$, $\min\limits_{m=1,2}\frac{\left|[\mathrm{Full}(\mathbf{H})]^{\dagger}\mathbf{h}_m\right|^2-\frac{1}{P}}{\chi_m}=\frac{\gamma(\mathrm{Full}(\mathbf{H}),\mathbf{H})-\frac{1}{P}}{\chi_{m_{\max}}}$. Therefore, $\min\limits_{m=1,2}\frac{\left|[\mathrm{Full}(\mathbf{H})]^{\dagger}\mathbf{h}_m\right|^2-\frac{1}{P}}{\chi_m}\ge\frac{\gamma(\mathrm{Full}(\mathbf{H}),\mathbf{H})-\frac{1}{P}}{\chi_{m_{\max}}}$, and

$$\log\frac{1}{1-p}\le 2(t-1)\log\frac{\max\limits_{m=1,2}\chi_m}{\gamma(\mathrm{Full}(\mathbf{H}),\mathbf{H})-\frac{1}{P}}.\tag{26}$$

Define $\mathcal{H}_4=\{\mathbf{H}:\mathbf{H}\in\mathtt{C}^{t\times 2},\chi_1\ge\chi_2\}$ and $\mathcal{H}_5=\{\mathbf{H}:\mathbf{H}\in\mathtt{C}^{t\times 2},\chi_1<\chi_2\}$. Substituting (26) and (17) into I_3 yields

$$\begin{aligned}I_3 &\le \frac{C_{15}}{P^2}\int_{\mathbf{H}\in\mathcal{H}_1\cap\mathcal{H}_4}\left[\frac{1}{\chi_1}+\frac{1}{\chi_2}\right]^2\log\frac{\chi_1}{\gamma(\mathrm{Full}(\mathbf{H}),\mathbf{H})-\frac{1}{P}}f_{\mathbf{H}}(\mathbf{H})\,\mathrm{d}\mathbf{H}\\ &\quad+\frac{C_{15}}{P^2}\int_{\mathbf{H}\in\mathcal{H}_1\cap\mathcal{H}_5}\left[\frac{1}{\chi_1}+\frac{1}{\chi_2}\right]^2\log\frac{\chi_2}{\gamma(\mathrm{Full}(\mathbf{H}),\mathbf{H})-\frac{1}{P}}f_{\mathbf{H}}(\mathbf{H})\,\mathrm{d}\mathbf{H}\\ &=\frac{2C_{15}}{P^2}\int_{\mathbf{H}\in\mathcal{H}_1\cap\mathcal{H}_4}\left[\frac{1}{\chi_1}+\frac{1}{\chi_2}\right]^2\log\frac{\chi_1}{\gamma(\mathrm{Full}(\mathbf{H}),\mathbf{H})-\frac{1}{P}}f_{\mathbf{H}}(\mathbf{H})\,\mathrm{d}\mathbf{H},\end{aligned}$$

where $C_{15}=2(t-1)^3C_3$. For any $\mathbf{H}\in\mathcal{H}_1\cap\mathcal{H}_4$, $\left[\frac{1}{\chi_1}+\frac{1}{\chi_2}\right]^2\le\left[\frac{1}{\chi_2}+\frac{1}{\chi_2}\right]^2=\frac{4}{\chi_2^2}$. Therefore, it follows that

$$I_3\le\frac{C_{16}}{P^2}\int_{\mathbf{H}\in\mathcal{H}_1\cap\mathcal{H}_4}\frac{1}{\chi_2^2}\log\frac{\chi_1}{\gamma(\mathrm{Full}(\mathbf{H}),\mathbf{H})-\frac{1}{P}}f_{\mathbf{H}}(\mathbf{H})\,\mathrm{d}\mathbf{H},\tag{27}$$

where $C_{16} = 8C_{15}$.

Define $\mathcal{H}_6 = \left\{\mathbf{H} : \mathbf{H} \in \mathcal{H}_1 \bigcap \mathcal{H}_4, \chi_2 \le \left|\mathbf{h}_1^\dagger \mathbf{h}_2\right|\right\}$ and $\mathcal{H}_7 = \left\{\mathbf{H} : \mathbf{H} \in \mathcal{H}_1 \bigcap \mathcal{H}_4, \chi_2 > \left|\mathbf{h}_1^\dagger \mathbf{h}_2\right|\right\}$. With such notations, $\gamma\left(\mathrm{Full}\left(\mathbf{H}\right), \mathbf{H}\right)$ in [6, Theorem 2] can be rewritten as

$$\gamma\left(\mathrm{Full}\left(\mathbf{H}\right), \mathbf{H}\right) = \begin{cases} \chi_2, & \mathbf{H} \in \mathcal{H}_6, \\ \frac{\chi_2}{1+\beta^2}, & \mathbf{H} \in \mathcal{H}_7, \end{cases}$$

where $\beta = \frac{\sqrt{\chi_2} - \sqrt{\chi_1 \theta}}{\sqrt{\chi_1 - \chi_1 \theta}}$ and $\theta = \frac{\left|\mathbf{h}_1^\dagger \mathbf{h}_2\right|^2}{\chi_1 \chi_2}$. Then, the upper bound on I_3 in (27) can be further deduced as

$$\begin{aligned} I_3 &\le \frac{C_{16}}{P^2} \int_{\mathbf{H}\in\mathcal{H}_6} \frac{1}{\chi_2^2} \log \frac{\chi_1}{\chi_2 - \frac{1}{P}} f_{\mathbf{H}}(\mathbf{H}) \mathrm{d}\mathbf{H} + \frac{C_{16}}{P^2} \int_{\mathbf{H}\in\mathcal{H}_7} \frac{1}{\chi_2^2} \log \frac{\chi_1}{\frac{\chi_2}{1+\beta^2} - \frac{1}{P}} f_{\mathbf{H}}(\mathbf{H}) \mathrm{d}\mathbf{H} \\ &= \underbrace{\frac{C_{16}}{P^2} \int_{\mathbf{H}\in\mathcal{H}_6 \cup \mathcal{H}_7} \frac{1}{\chi_2^2} \left(\log \chi_1\right) f_{\mathbf{H}}(\mathbf{H}) \mathrm{d}\mathbf{H}}_{=I_{3,1}} + \underbrace{\frac{C_{16}}{P^2} \int_{\mathbf{H}\in\mathcal{H}_6} \frac{1}{\chi_2^2} \log \frac{1}{\chi_2 - \frac{1}{P}} f_{\mathbf{H}}(\mathbf{H}) \mathrm{d}\mathbf{H}}_{=I_{3,2}} \\ &+ \underbrace{\frac{C_{16}}{P^2} \int_{\mathbf{H}\in\mathcal{H}_7} \frac{1}{\chi_2^2} \log \left(1+\beta^2\right) f_{\mathbf{H}}(\mathbf{H}) \mathrm{d}\mathbf{H}}_{=I_{3,3}} + \underbrace{\frac{C_{16}}{P^2} \int_{\mathbf{H}\in\mathcal{H}_7} \frac{1}{\chi_2^2} \log \frac{1}{\chi_2 - \frac{1+\beta^2}{P}} f_{\mathbf{H}}(\mathbf{H}) \mathrm{d}\mathbf{H}}_{=I_{3,4}}. \end{aligned}$$

Since $\{\mathcal{H}_6 \cup \mathcal{H}_7\} = \{\mathcal{H}_1 \cap \mathcal{H}_4\} \subseteq \mathcal{H}_1 \subseteq \mathcal{H}_0$ and $\log(x) \le x$ for $x > 0$, the upper bound on $I_{3,1}$ is derived as

$$\begin{aligned} I_{3,1} &\le \frac{C_{16}}{P^2} \int_{\frac{1}{P}}^{\infty} \int_{\frac{1}{P}}^{\infty} \frac{\chi_1}{\chi_2^2} \left[\frac{\chi_1^{t-1} e^{-\chi_1}}{(t-1)!}\right] \left[\frac{\chi_2^{t-1} e^{-\chi_2}}{(t-1)!}\right] \mathrm{d}\chi_{\mathbf{1}} \mathrm{d}\chi_{\mathbf{2}} \\ &= \frac{C_{16}}{P^2 \left[(t-1)!\right]^2} \int_{\frac{1}{P}}^{\infty} \chi_1^t e^{-\chi_1} \mathrm{d}\chi_1 \int_{\frac{1}{P}}^{\infty} \chi_2^{t-3} e^{-\chi_2} \mathrm{d}\chi_2 \\ &\le \frac{C_{16}}{P^2 \left[(t-1)!\right]^2} \int_0^{\infty} \chi_1^t e^{-\chi_1} \mathrm{d}\chi_1 \int_{\frac{1}{P}}^{\infty} \chi_2^{t-3} e^{-\chi_2} \mathrm{d}\chi_2 \\ &= \frac{C_{17}}{P^2} \int_{\frac{1}{P}}^{\infty} \chi_2^{t-3} e^{-\chi_2} \mathrm{d}\chi_2, \end{aligned}$$

where $C_{17} = \frac{tC_{16}}{(t-1)!}$. When $t \ge 3$, using (22), the bound on $I_{3,1}$ becomes

$$I_{3,1} \le \frac{C_{17}}{P^2} \Gamma\left(t-2, \frac{1}{P}\right) \le C_{18} e^{-\frac{1}{P}} \left[\frac{1}{P^2} + \frac{1}{P^{t-1}}\right], \tag{28}$$

where $C_{18} = (t-2)! C_{17}$. When $t = 2$, we obtain

$$I_{3,1} \le \frac{C_{17}}{P^2} \int_{\frac{1}{P}}^{\infty} \frac{e^{-\chi_2}}{\chi_2} \mathrm{d}\chi_2 = \frac{C_{17}}{P^2} \mathtt{E}_1\left(\frac{1}{P}\right) \le C_{17} \frac{e^{-\frac{1}{P}}}{P^2} \log(1+P). \tag{29}$$

Then, from (28) and (29), for any $t \geq 2$, the upper bound for $I_{3,1}$ can be

$$\begin{aligned} I_{3,1} &\leq C_{18} e^{-\frac{1}{P}} \left[\frac{1}{P^2} + \frac{1}{P^{t-1}} \right] \times \mathbf{1}_{t\geq 3} + C_{17} \frac{e^{-\frac{1}{P}}}{P^2} \log(1+P) \times \mathbf{1}_{t=2} \\ &\leq C_{19} e^{-\frac{1}{P}} \left[\frac{1}{P} + \frac{1}{P^{2t}} + \frac{\log(1+P)}{P} \right], \end{aligned} \tag{30}$$

where $C_{19} = 2C_{18} \times \mathbf{1}_{t\geq 3} + C_{17} \times \mathbf{1}_{t=2}$. The last inequality can be verified by comparing both cases when $0 < P \leq 1$ and $P > 1$.

For $I_{3,2}$, since $\mathcal{H}_6 \subseteq \mathcal{H}_0$, its upper bound can be

$$\begin{aligned} I_{3,2} &\leq \frac{C_{16}}{P^2} \int_{\frac{1}{P}}^{\infty} \left[\frac{\chi_1^{t-1} e^{-\chi_1}}{(t-1)!} \right] \mathrm{d}\chi_1 \int_{\frac{1}{P}}^{\infty} \frac{1}{\chi_2^2} \log \frac{1}{\chi_2 - \frac{1}{P}} \left[\frac{\chi_2^{t-1} e^{-\chi_2}}{(t-1)!} \right] \mathrm{d}\chi_2 \\ &\leq \frac{C_{16}}{(t-1)! P^2} \int_0^{\infty} \frac{\chi_1^{t-1} e^{-\chi_1}}{(t-1)!} \mathrm{d}\chi_1 \int_{\frac{1}{P}}^{\infty} \chi_2^{t-3} e^{-\chi_2} \log \frac{1}{\chi_2 - \frac{1}{P}} \mathrm{d}\chi_2 \\ &= C_{20} \frac{e^{-\frac{1}{P}}}{P^2} \int_0^{\infty} \left(\log \frac{1}{\lambda_2} \right) \left(\lambda_2 + \frac{1}{P} \right)^{t-3} e^{-\lambda_2} \mathrm{d}\lambda_2, \end{aligned} \tag{31}$$

where $C_{20} = \frac{C_{16}}{(t-1)!}$ and the last equality arises from replacing $\chi_2 - \frac{1}{P}$ by λ_2. When $t \geq 4$, with the help of (22), we get

$$\begin{aligned} I_{3,2} &\leq C_{20} \frac{e^{-\frac{1}{P}}}{P^2} \int_0^{\frac{1}{P}} \left(\log \frac{1}{\lambda_2} \right) \left(\lambda_2 + \frac{1}{P} \right)^{t-3} e^{-\lambda_2} \mathrm{d}\lambda_2 + C_{20} \frac{e^{-\frac{1}{P}}}{P^2} \int_{\frac{1}{P}}^{\infty} \left(\log \frac{1}{\lambda_2} \right) \left(\lambda_2 + \frac{1}{P} \right)^{t-3} e^{-\lambda_2} \mathrm{d}\lambda_2 \\ &\leq C_{20} \frac{e^{-\frac{1}{P}}}{P^2} \int_0^{\frac{1}{P}} \left(\log \frac{1}{\lambda_2} \right) \left(\frac{1}{P} + \frac{1}{P} \right)^{t-3} \mathrm{d}\lambda_2 + C_{20} \frac{e^{-\frac{1}{P}}}{P^2} \int_{\frac{1}{P}}^{\infty} \frac{1}{\lambda_2} \left(\lambda_2 + \lambda_2 \right)^{t-3} e^{-\lambda_2} \mathrm{d}\lambda_2 \\ &\leq \frac{2^{t-3} C_{20} e^{-\frac{1}{P}}}{P^{t-1}} \int_0^{\frac{1}{P}} \log \frac{1}{\lambda_2} \mathrm{d}\lambda_2 + \frac{2^{t-3} C_{20}}{P^2} \int_{\frac{1}{P}}^{\infty} \lambda_2^{t-4} e^{-\lambda_2} \mathrm{d}\lambda_2 \\ &= \frac{C_{21} e^{-\frac{1}{P}}}{P^{t-1}} \left[\frac{1}{P} + \frac{\log P}{P} \right] + \frac{C_{21}}{P^2} \Gamma \left(t-3, \frac{1}{P} \right) \\ &\leq \frac{C_{21} e^{-\frac{1}{P}}}{P^{t-1}} \left[\frac{1}{P} + 1 \right] + \frac{(t-3)! C_{21} e^{-\frac{1}{P}}}{P^2} \left[1 + \frac{1}{P^{t-4}} \right] \\ &\leq C_{22} e^{-\frac{1}{P}} \left[\frac{1}{P} + \frac{1}{P^{2t}} \right], \end{aligned} \tag{32}$$

where $C_{21} = 2^{t-3}C_{20}$ and $C_{22} = 2 \times (t-3)!C_{21} + 2C_{21}$. When $t = 3$, (31) becomes

$$\begin{aligned}
I_{3,2} &\leq C_{20}\frac{e^{-\frac{1}{P}}}{P^2}\int_0^{\frac{1}{P}}\left(\log\frac{1}{\lambda_2}\right)e^{-\lambda_2}\mathrm{d}\lambda_2 + C_{20}\frac{e^{-\frac{1}{P}}}{P^2}\int_{\frac{1}{P}}^{\infty}\left(\log\frac{1}{\lambda_2}\right)e^{-\lambda_2}\mathrm{d}\lambda_2 \\
&\leq C_{20}\frac{e^{-\frac{1}{P}}}{P^2}\int_0^{\frac{1}{P}}\log\frac{1}{\lambda_2}\mathrm{d}\lambda_2 + \frac{C_{20}}{P^2}\int_{\frac{1}{P}}^{\infty}\frac{e^{-\lambda_2}}{\lambda_2}\mathrm{d}\lambda_2 \\
&\leq C_{20}\frac{e^{-\frac{1}{P}}}{P^2}\left[\frac{1}{P} + \frac{\log P}{P}\right] + \frac{C_{20}}{P^2}\mathtt{E}_1\left(\frac{1}{P}\right) \\
&\leq C_{20}\frac{e^{-\frac{1}{P}}}{P^2}\left[\frac{1}{P} + 1\right] + \frac{C_{20}}{P^2}e^{-\frac{1}{P}}\log(1+P) \\
&\leq C_{23}e^{-\frac{1}{P}}\left[\frac{1}{P} + \frac{1}{P^{2t}} + \frac{\log(1+P)}{P}\right], \qquad (33)
\end{aligned}$$

where $C_{23} = 3C_{20}$. When $t = 2$, since $\frac{1}{\lambda_2+\frac{1}{P}} \leq P$, $I_{3,2} \leq C_{20}\frac{e^{-\frac{1}{P}}}{P}\int_0^{\infty}\log\frac{1}{\lambda_2}e^{-\lambda_2}\mathrm{d}\lambda_2$. Following the same steps in (33), $I_{3,2}$ can be bounded by

$$I_{3,2} \leq C_{24}e^{-\frac{1}{P}}\left[\frac{1}{P} + \frac{1}{P^{2t}} + \frac{\log(1+P)}{P}\right], \qquad (34)$$

where $C_{24} = 3C_{20}$. Based on (32), (33) and (34), the upper bound on $I_{3,2}$ for any $t \geq 2$ is

$$I_{3,2} \leq C_{25}e^{-\frac{1}{P}}\left[\frac{1}{P} + \frac{1}{P^{2t}} + \frac{\log(1+P)}{P}\right], \qquad (35)$$

where $C_{25} = C_{22} \times \mathbf{1}_{t\geq 4} + C_{23} \times \mathbf{1}_{t=3} + C_{24} \times \mathbf{1}_{t=2}$.

In $I_{3,3}$, since $0 \leq \beta \leq 1$, $\log(1+\beta^2) \leq \log 2 < 1$. Similar to the derivation for $I_{3,1}$, the bound on $I_{3,3}$ is obtained as

$$I_{3,3} \leq C_{26}e^{-\frac{1}{P}}\left[\frac{1}{P} + \frac{1}{P^{2t}} + \frac{\log(1+P)}{P}\right], \qquad (36)$$

where $C_{26} = \frac{2C_{16}}{t-1} \times \mathbf{1}_{t\geq 3} + \frac{C_{16}}{(t-1)!} \times \mathbf{1}_{t=2}$.

For $I_{3,4}$, since χ_1, χ_2 and θ are mutually independent, its upper bound can be derived as

$$\begin{aligned}
I_{3,4} &\leq \frac{C_{16}}{P^2}\int_{\{\chi_1,\chi_2,\theta\}\in\mathcal{H}_7'}\frac{1}{\chi_2^2}\left(\log\frac{1}{\chi_2 - \frac{1+\beta^2}{P}}\right)f_{\chi_1}(\chi_1)f_{\chi_2}(\chi_2)f_{\theta}(\theta)\mathrm{d}\chi_1\mathrm{d}\chi_2\mathrm{d}\theta \\
&= \frac{C_{27}}{P^2}\int_{\{\chi_1,\chi_2,\theta\}\in\mathcal{H}_7'}\left(\log\frac{1}{\chi_2 - \frac{1+\beta^2}{P}}\right)\chi_1^{t-1}e^{-\chi_1}\chi_2^{t-3}e^{-\chi_2}(1-\theta)^{t-2}\mathrm{d}\chi_1\mathrm{d}\chi_2\mathrm{d}\theta,
\end{aligned}$$

where $C_{27} = \frac{C_{16}}{(t-1)!(t-2)!}$ and $\mathcal{H}_7'$ is a transformed version of the pre-defined $\mathcal{H}_7$ with respect to χ_1, χ_2 and θ. The PDF of θ is given by $f_\theta(\theta) = (t-1)(1-\theta)^{t-2}$ for $0 \leq \theta \leq 1$ [9]. By changing the integration variables from (χ_1, χ_2, θ) into (β, χ_2, θ), we obtain the Jacobian of the transformation as $\left|\frac{\partial(\chi_1,\chi_2,\theta)}{\partial(\beta,\chi_2,\theta)}\right| = \left|\frac{\partial\chi_1}{\partial\beta}\right|$. For any $\mathbf{H} \in \mathcal{H}_7'$, $\beta = \frac{\sqrt{\chi_2}-\sqrt{\chi_1\theta}}{\sqrt{\chi_1-\chi_1\theta}}$, $\chi_1 = \frac{\chi_2}{\left(\sqrt{\theta}+\beta\sqrt{1-\theta}\right)^2}$ and $\left|\frac{\partial\chi_1}{\partial\beta}\right| = \frac{2\sqrt{1-\theta}\chi_2}{\left(\sqrt{\theta}+\beta\sqrt{1-\theta}\right)^3}$. Therefore, $I_{3,4}$ can be bounded by

$$\begin{aligned}
I_{3,4} \leq& \frac{C_{28}}{P^2}\int_{\{\beta,\chi_2,\theta\}\in\mathcal{H}_7''}\left(\log\frac{1}{\chi_2-\frac{1+\beta^2}{P}}\right)\left[\frac{\chi_2}{(\sqrt{\theta}+\beta\sqrt{1-\theta})^2}\right]^{t-1} e^{-\frac{\chi_2}{(\sqrt{\theta}+\beta\sqrt{1-\theta})^2}}\chi_2^{t-3}e^{-\chi_2} \\
&\times(1-\theta)^{t-2}\frac{\sqrt{1-\theta}\chi_2}{\left(\sqrt{\theta}+\beta\sqrt{1-\theta}\right)^3}\mathrm{d}\chi_2\mathrm{d}\beta\mathrm{d}\theta \\
=& \frac{C_{28}}{P^2}\int_{\{\beta,\chi_2,\theta\}\in\mathcal{H}_7''}\left(\log\frac{1}{\chi_2-\frac{1+\beta^2}{P}}\right)\frac{\chi_2^{2t-3}e^{-\chi_2}(1-\theta)^{t-\frac{3}{2}}}{\left(\sqrt{\theta}+\beta\sqrt{1-\theta}\right)^{2t+1}}e^{-\frac{\chi_2}{(\sqrt{\theta}+\beta\sqrt{1-\theta})^2}}\mathrm{d}\chi_2\mathrm{d}\beta\mathrm{d}\theta \\
\leq& \frac{C_{28}}{P^2}\int_{\{\beta,\chi_2,\theta\}\in\mathcal{H}_7''}\left(\log\frac{1}{\chi_2-\frac{1+\beta^2}{P}}\right)\frac{\chi_2^{2t-3}e^{-\chi_2}}{\left(\sqrt{\theta}+\beta\sqrt{1-\theta}\right)^{2t+1}}e^{-\frac{\chi_2}{(\sqrt{\theta}+\beta\sqrt{1-\theta})^2}}\mathrm{d}\chi_2\mathrm{d}\beta\mathrm{d}\theta,
\end{aligned}$$

where $C_{28} = 2C_{27}$ and $\mathcal{H}_7''$ is a transformed version of $\mathcal{H}_7'$ with respect to β,χ_2 and θ. By replacing $\chi_2 - \frac{1+\beta^2}{P}$ by χ, $\left|\frac{\partial(\beta,\chi_2,\theta)}{\partial(\beta,\chi,\theta)}\right| = \left|\frac{\partial\chi_2}{\partial\chi}\right| = 1$, then, $I_{3,4}$ is further bounded by

$$\begin{aligned}
I_{3,4} \leq& \frac{C_{28}}{P^2}\int_{\{\beta,\chi,\theta\}\in\mathcal{H}_7'''}\left(\log\frac{1}{\chi}\right)\frac{e^{-\frac{\chi+\frac{1+\beta^2}{P}}{(\sqrt{\theta}+\beta\sqrt{1-\theta})^2}}}{\left(\sqrt{\theta}+\beta\sqrt{1-\theta}\right)^{2t+1}}\left[\chi+\frac{1+\beta^2}{P}\right]^{2t-3}e^{-\chi-\frac{1+\beta^2}{P}}\mathrm{d}\chi\mathrm{d}\beta\mathrm{d}\theta \\
\leq& \frac{C_{28}}{P^2}\int_{\{\beta,\chi,\theta\}\in\mathcal{H}_7'''}\left(\log\frac{1}{\chi}\right)\frac{e^{-\frac{\chi+\frac{1+\beta^2}{P}}{(\sqrt{\theta}+\beta\sqrt{1-\theta})^2}}}{\left(\sqrt{\theta}+\beta\sqrt{1-\theta}\right)^{2t+1}}\left[\chi+\frac{1+\beta^2}{P}\right]^{2t-3}e^{-\chi-\frac{1}{P}}\mathrm{d}\chi\mathrm{d}\beta\mathrm{d}\theta, \quad (37)
\end{aligned}$$

where $\mathcal{H}_7'''$ is a transformed version of $\mathbf{H}_7''$ with respect to β,χ and θ. Letting $\phi = \frac{\chi+\frac{1+\beta^2}{P}}{\left(\sqrt{\theta}+\beta\sqrt{1-\theta}\right)^2}$, $\left|\frac{\partial(\chi,\beta,\theta)}{\partial(\chi,\beta,\phi)}\right| = \left|\frac{\partial\theta}{\partial\phi}\right|$. Since $\frac{\sqrt{\chi+\frac{1+\beta^2}{P}}}{\sqrt{\phi}} = \sqrt{\theta}+\beta\sqrt{1-\theta}$, $\left|\frac{\partial\theta}{\partial\phi}\right| = \frac{\phi^{-\frac{3}{2}}\sqrt{\chi+\frac{1+\beta^2}{P}}}{\left|\frac{1}{\sqrt{\theta}}-\frac{\beta}{\sqrt{1-\theta}}\right|}$. For any $\mathbf{H} \in \mathcal{H}_7'''$, $\chi_1 \geq \chi_2$, thus, $\phi = \frac{\chi_1}{\chi_2} \geq 1$ and $0 \leq \sqrt{\theta}+\beta\sqrt{1-\theta} \leq 1$. Then, $0 \leq \beta \leq \frac{1-\sqrt{\theta}}{\sqrt{1-\theta}}$. Hence, $\frac{1}{\sqrt{\theta}}-\frac{\beta}{\sqrt{1-\theta}} \geq \frac{1}{\sqrt{\theta}}-\frac{1}{\sqrt{1-\theta}}\times\frac{1-\sqrt{\theta}}{\sqrt{1-\theta}} = \frac{1}{(1+\sqrt{\theta})\sqrt{\theta}} > 0$. Therefore, $\left|\frac{\partial\theta}{\partial\phi}\right| \leq \phi^{-\frac{3}{2}}\sqrt{\chi+\frac{1+\beta^2}{P}}(1+\sqrt{\theta})\sqrt{\theta} \leq 2\phi^{-\frac{3}{2}}\sqrt{\chi+\frac{1+\beta^2}{P}}$ due to $0 \leq \theta \leq 1$. Moreover, since $\mathcal{H}_7''' \subseteq \{(\beta,\chi,\phi) : 0 \leq \beta \leq 1, \chi > 0, \phi > 0\}$, the upper bound in (37) becomes

$$
\begin{aligned}
I_{3,1,4} &\leq 2C_{28}\frac{e^{-\frac{1}{P}}}{P^2}\int_{\{\beta,\chi,\phi\}\in\mathcal{H}_7'''}\left(\log\frac{1}{\chi}\right)e^{-\phi}\left[\chi+\frac{1+\beta^2}{P}\right]^{t-3}\phi^{t-1}e^{-\chi}\mathrm{d}\chi\mathrm{d}\beta\mathrm{d}\phi \\
&\leq 2C_{28}\frac{e^{-\frac{1}{P}}}{P^2}\int_0^\infty\int_0^1\int_0^\infty\left(\log\frac{1}{\chi}\right)e^{-\phi}\left[\chi+\frac{1+\beta^2}{P}\right]^{t-3}\phi^{t-1}e^{-\chi}\mathrm{d}\chi\mathrm{d}\beta\mathrm{d}\phi \\
&= 2C_{28}\frac{e^{-\frac{1}{P}}}{P^2}\left[\int_0^\infty\phi^{t-1}e^{-\phi}\mathrm{d}\phi\right]\int_0^\infty\int_0^1\left(\log\frac{1}{\chi}\right)\left[\chi+\frac{1+\beta^2}{P}\right]^{t-3}e^{-\chi}\mathrm{d}\chi\mathrm{d}\beta \\
&\leq C_{29}\frac{e^{-\frac{1}{P}}}{P^2}\int_0^\infty\int_0^1\left(\log\frac{1}{\chi}\right)\left[\chi+\frac{1+\beta^2}{P}\right]^{t-3}e^{-\chi}\mathrm{d}\chi\mathrm{d}\beta, \qquad (38)
\end{aligned}
$$

where $C_{29}=2(t-1)!C_{28}$. When $t\geq 4$, since $\frac{1+\beta^2}{P}\leq\frac{2}{P}$ due to $0\leq\beta\leq 1$, we obtain

$$
\begin{aligned}
I_{3,4} &\leq C_{29}\frac{e^{-\frac{1}{P}}}{P^2}\int_0^\infty\left(\log\frac{1}{\chi}\right)\left[\chi+\frac{2}{P}\right]^{t-3}e^{-\chi}\mathrm{d}\chi \\
&= C_{29}\frac{e^{-\frac{1}{P}}}{P^2}\left[\int_0^{\frac{2}{P}}\left(\log\frac{1}{\chi}\right)\left[\chi+\frac{2}{P}\right]^{t-3}e^{-\chi}\mathrm{d}\chi+\int_{\frac{2}{P}}^\infty\left(\log\frac{1}{\chi}\right)\left[\chi+\frac{2}{P}\right]^{t-3}e^{-\chi}\mathrm{d}\chi\right] \\
&\leq C_{29}\frac{e^{-\frac{1}{P}}}{P^2}\left[\int_0^{\frac{2}{P}}\left(\log\frac{1}{\chi}\right)\left[\frac{2}{P}+\frac{2}{P}\right]^{t-3}\mathrm{d}\chi+\int_{\frac{2}{P}}^\infty\frac{1}{\chi}\left[\chi+\chi\right]^{t-3}e^{-\chi}\mathrm{d}\chi\right] \\
&\leq 4^{t-3}C_{29}\frac{e^{-\frac{1}{P}}}{P^{t-1}}\int_0^{\frac{2}{P}}\log\frac{1}{\chi}\mathrm{d}\chi+\frac{2^{t-3}C_{29}}{P^2}\int_{\frac{2}{P}}^\infty\chi^{t-4}e^{-\chi}\mathrm{d}\chi \\
&= C_{30}\frac{e^{-\frac{1}{P}}}{P^{t-1}}\left[\frac{2}{P}+\frac{2\log\frac{P}{2}}{P}\right]+\frac{C_{31}}{P^2}\Gamma\left(t-3,\frac{2}{P}\right) \\
&\leq C_{30}\frac{e^{-\frac{1}{P}}}{P^{t-1}}\left[\frac{2}{P}+1\right]+C_{31}(t-4)!\frac{e^{-\frac{2}{P}}}{P^2}\left[1+\left(\frac{2}{P}\right)^{t-4}\right] \\
&\leq C_{30}\frac{e^{-\frac{1}{P}}}{P^{t-1}}\left[\frac{2}{P}+1\right]+C_{32}\frac{e^{-\frac{1}{P}}}{P^2}\left[1+\left(\frac{2}{P}\right)^{t-4}\right] \\
&\leq C_{33}e^{-\frac{1}{P}}\left[\frac{1}{P}+\frac{1}{P^{2t}}\right], \qquad (39)
\end{aligned}
$$

where $C_{30}=4^{t-3}C_{29}$, $C_{31}=2^{t-3}C_{29}$, $C_{32}=(t-4)!C_{31}$, and $C_{33}=2C_{30}+2C_{32}$. When $t=3$, (38) is simplified to be $I_{3,4}\leq C_{29}\frac{e^{-\frac{1}{P}}}{P^2}\int_0^\infty\left(\log\frac{1}{\chi}\right)e^{-\chi}\mathrm{d}\chi$. Similar to (33), the upper bound can be derived as

$$
I_{3,4}\leq C_{34}e^{-\frac{1}{P}}\left[\frac{1}{P}+\frac{1}{P^{2t}}+\frac{\log(1+P)}{P}\right], \qquad (40)
$$

where $C_{34} = 3C_{29}$. When $t = 2$, since $\frac{1}{\chi + \frac{1+\beta^2}{P}} \leq \frac{P}{1+\beta^2} \leq P$, $I_{3,4} \leq C_{29} \frac{e^{-\frac{1}{P}}}{P} \int_0^\infty \log \frac{1}{\chi} e^{-\chi} \mathrm{d}\chi$. Still applying the same derivation in (40), we obtain

$$I_{3,4} \leq C_{35} e^{-\frac{1}{P}} \left[\frac{1}{P} + \frac{1}{P^{2t}} + \frac{\log(1+P)}{P} \right], \tag{41}$$

where $C_{35} = 3C_{29}$. Combining bounds derived in (39), (40) and (41), the bound on $I_{3,4}$ for any $t \geq 2$ is

$$I_{3,4} \leq C_{36} e^{-\frac{1}{P}} \left[\frac{1}{P} + \frac{1}{P^{2t}} + \frac{\log(1+P)}{P} \right], \tag{42}$$

where $C_{36} = C_{33} \times \mathbf{1}_{t \geq 4} + C_{34} \times \mathbf{1}_{t=3} + C_{35} \times \mathbf{1}_{t=2}$. Based on (30), (35), (36) and (42), I_3 is upper-bounded by

$$I_3 \leq C_{37} e^{-\frac{1}{P}} \left[\frac{1}{P} + \frac{1}{P^{2t}} + \frac{\log(1+P)}{P} \right], \tag{43}$$

where $C_{37} = C_{19} + C_{25} + C_{26} + C_{36}$. Finally, from (20), (25) and (43), we get the upper bound in (13), where $C_0 = C_7 + C_{14} + C_{37}$. ■

Remark 2: We mainly focus on showing how the number of average feedback bits for $\mathsf{Q}_{\mathrm{VLQ}}$ changes with P. Therefore, it is beyond the scope of this paper to find the tightest bound, i.e., the smallest value for C_0.

Remark 3: From (13), it can be seen that in the medium and high regions for P, the derived upper bound on average feedback rate is dominated by $e^{-\frac{1}{P}} \left[\frac{1}{P} + \frac{\log(1+P)}{P} \right]$; in the low region for P, it is dominated by $\frac{e^{-\frac{1}{P}}}{P^{2t}}$. Moreover, the upper bound will approach zero when $P \to \infty$ and $P \to 0$. The average feedback rate also behaves like this. This can be intuitively interpreted as follows: when $P \to \infty$, any vector in the codebook will not cause an outage event, while when $P \to 0$, any vector will result in outage. According to the encoding rule of $\mathsf{Q}_{\mathrm{VLQ}}$, only empty codewords will be fed back in both situations. Thus, the average feedback rate approaches zero.

VI. Numerical Simulations

In this section, we perform numerical simulations to verify the theoretical results for the outage probability and the average feedback rate.

For each value of P, a sufficiently large number of channel realizations will be generated in order to observe 1000 outage events. In the pseudo-code presented below, `OUT` stands for the simulated outage probability and `R` refers to the simulated average feedback rate. For each channel realization, whether the full-CSI case could prevent outage will be checked firstly. If not,

Simulation Procedure:

Initialization: given P, $\texttt{Loop} = 0$, $\texttt{OUT} = 0$, $\texttt{R} = 0$;
while $\texttt{OUT} < 1000$
    $\texttt{Index} = 0$;
    $\texttt{Loop} = \texttt{Loop} + 1$;
    generate a realization of $\mathbf{H}$;
    if $\gamma\left(\text{Full}\left(\mathbf{H}\right), \mathbf{H}\right) < \frac{1}{P}$
        $\texttt{OUT} = \texttt{OUT} + 1$;
    else
        randomly generate $\mathbf{x} \in \mathcal{X}$;
        while $\gamma\left(\mathbf{x}, \mathbf{H}\right) < \frac{1}{P}$
            randomly generate $\mathbf{y} \in \mathcal{X}$;
            $\mathbf{x} = \mathbf{y}$;
            $\texttt{Index} = \texttt{Index} + 1$;
        end
    end
    $\texttt{R} = \texttt{R} + \lfloor \log_2(1 + \texttt{Index}) \rfloor$;
end
return $\texttt{OUT} = \frac{\texttt{OUT}}{\texttt{Loop}}$, $\texttt{R} = \frac{\texttt{R}}{\texttt{Loop}}$.

an outage event is declared; otherwise, a random unit-normal vector will be generated repeatedly until one that allows the channel realization to avoid outage is found. Finally, the simulated outage probability is computed as 1000 divided by the number of all channel realizations, while the simulated feedback rate is the average number of feedback bits. In our simulations, no endless iteration is detected, which is equivalent to say that as long as the channel state is able to avoid outage in the full-CSI case, a codeword that can also result in non-outage will be eventually found in the randomly-generated codebook.

In Figs. 1 and 2, we plot the simulated outage probabilities and average feedback rates for $t = 2$ and 3. The horizontal axis represents P in decibels. It can be observed that the average feedback rates in both cases will decrease towards zero when P increases towards infinity or

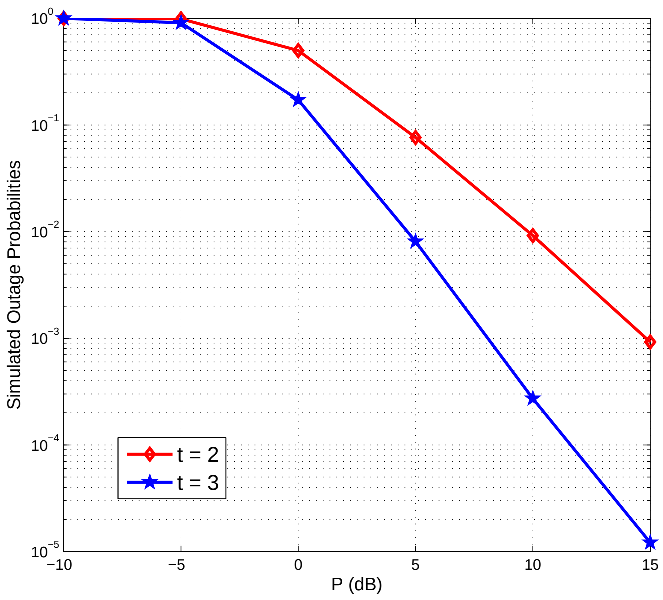

Fig. 1. Simulated outage probabilities when $t = 2, 3$.

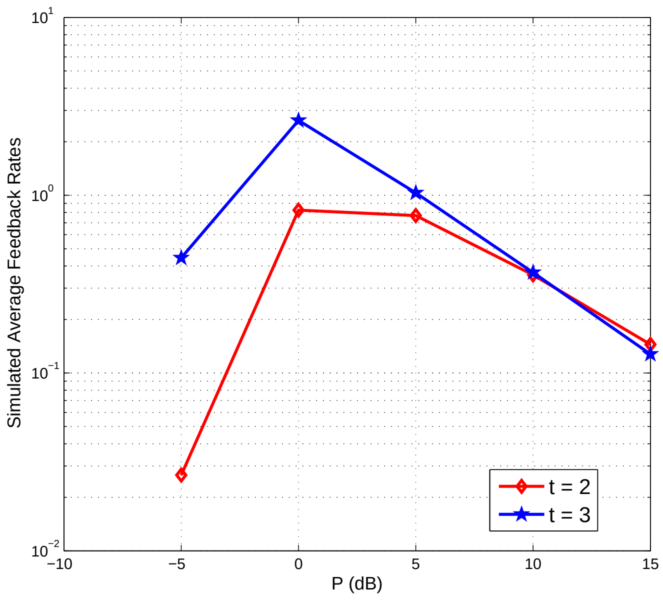

Fig. 2. Simulated average feedback rates when $t = 2, 3$.

decreases to zero. Furthermore, the average feedback rates are small for all P. When $t = 2$, the average feedback rates for all P are no larger than 1 bit per channel state; when $t = 3$, when $P \leq -5$ dB or $P \geq 10$ dB, the average feedback rates are as low as 0.5 bits per channel state.

VII. Conclusions and Future Work

In this paper, we have proved that in the two-user multicast network, the proposed VLQ can achieve the full-CSI outage probability with a low average feedback rate. In the future, we intend to work on a distributed quantizer for the multicast network. In this scenario, each receiver only feedbacks its local channel information and no node can acquire the full CSI. We still aim to approach the full-CSI outage probability at the cost of a finite average feedback rate.

Appendix A - Proof of Lemma 1

Proof: We use the following lemma, the proof of which is given in Appendix D.

Lemma 4. *For unit-normal complex vectors* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{C}^{t\times 1}$, *we have*

$$\left| |\mathbf{u}^\dagger \mathbf{v}|^2 - |\mathbf{u}^\dagger \mathbf{w}|^2 \right| \leq \sqrt{1 - |\mathbf{v}^\dagger \mathbf{w}|^2}. \tag{44}$$

For any $\mathbf{H}$ satisfying $\gamma\left(\mathrm{Full}\left(\mathbf{H}\right), \mathbf{H}\right) > \frac{1}{P}$, let $\Delta_m = \left|\left[\mathrm{Full}\left(\mathbf{H}\right)\right]^\dagger \mathbf{h}_m\right|^2 - \frac{1}{P}$, where $0 < \frac{\Delta_m}{\chi_m} < 1$ for $m = 1, 2$. If $|\mathbf{x}^\dagger \mathrm{Full}\left(\mathbf{H}\right)|^2 \geq \Pi = 1 - \min\limits_{m=1,2} \left[\frac{\Delta_m}{\chi_m}\right]^2$, by applying (44) and letting $\mathbf{u} = \frac{\mathbf{h}_m}{|\mathbf{h}_m|}$, $\mathbf{v} = \mathbf{x}$, $\mathbf{w} = \mathrm{Full}\left(\mathbf{H}\right)$, we derive that

$$\begin{aligned}
&\left| \left|\frac{\mathbf{h}_m^\dagger}{|\mathbf{h}_m|}\mathbf{x}\right|^2 - \left|\frac{\mathbf{h}_m^\dagger}{|\mathbf{h}_m|}\mathrm{Full}\left(\mathbf{H}\right)\right|^2 \right| \leq \sqrt{1 - |\mathbf{x}^\dagger \mathrm{Full}\left(\mathbf{H}\right)|^2} \\
\Longrightarrow &\left|\frac{\mathbf{h}_m^\dagger}{|\mathbf{h}_m|}\mathbf{x}\right|^2 \geq \left|\frac{\mathbf{h}_m^\dagger}{|\mathbf{h}_m|}\mathrm{Full}\left(\mathbf{H}\right)\right|^2 - \sqrt{1 - |\mathbf{x}^\dagger \mathrm{Full}\left(\mathbf{H}\right)|^2} \\
\Longrightarrow &\left|\frac{\mathbf{h}_m^\dagger}{|\mathbf{h}_m|}\mathbf{x}\right|^2 \geq \frac{1}{P\chi_m} + \frac{\Delta_m}{\chi_m} - \sqrt{1 - \Pi} \geq \frac{1}{P\chi_m} \\
\Longrightarrow &\left|\mathbf{h}_m^\dagger \mathbf{x}\right|^2 \geq \frac{1}{P}.
\end{aligned}$$

Since $0 < \Pi < 1$, the proof is complete. ■

APPENDIX B - PROOF OF LEMMA 2

Proof: For $p = 0$, $\Phi = 0$ and the upper bound in (15) holds. Hence, suppose that $0 < p < 1$. Then,

$$\begin{aligned}\Phi &= \sum_{i=1}^{\infty} p^i (1-p) \lfloor \log_2(i+1) \rfloor \\ &= p(1-p) + \sum_{i=2}^{\infty} p^i (1-p) \lfloor \log_2(i+1) \rfloor \\ &\le p(1-p) + \sum_{i=2}^{\infty} p^i (1-p) \log_2(i+1) \\ &= p(1-p) + p(1-p) \sum_{i=1}^{\infty} p^i \log_2(i+2) \\ &= p(1-p) + p(1-p) \left[p \log_2 3 + \sum_{i=2}^{\infty} p^i \log_2(i+2) \right] \\ &\le p(1-p) + p(1-p) \left[p \log_2 3 + \frac{2}{\log 2} \sum_{i=1}^{\infty} p^i \log i \right]. \end{aligned} \tag{45}$$

We estimate the sum $\sum_{i=1}^{\infty} p^i \log i$ via the integral of the function $f(x) = e^{-\beta x} \log x$, where $0 < \beta \triangleq -\log p < \infty$. We calculate $f'(x) = e^{-\beta x} \left(\frac{1}{x} - \beta \log x \right)$, where f' represents the derivative of f. For $y \log y = \frac{1}{\beta}$, $f'(x) > 0$ for $1 \le x < y$, $f'(x) = 0$ for $x = y$, and $f'(x) < 0$ for $x > y$. The global maximum of f is thus $f(y)$. Since $y \log y = \frac{1}{\beta} > 0$, $y \ge 1$ must hold, which implies $f(y) = e^{-\beta y} \log y \le e^{-\beta} \log y \le e^{-\beta} y \log y = \frac{e^{-\beta}}{\beta}$. Let $j = \lfloor y \rfloor$. Then, $1 \le j \le y < j + 1$, and

$$\begin{aligned}\sum_{i=1}^{\infty} f(i) &= \mathbf{1}_{j \ge 2} \sum_{i=1}^{j-1} f(i) + f(j) + f(j+1) + \sum_{i=j+2}^{\infty} f(i) \\ &= \mathbf{1}_{j \ge 2} \sum_{i=1}^{j-1} \int_i^{i+1} f(i) \mathrm{d}x + f(j) + f(j+1) + \sum_{i=j+2}^{\infty} \int_{i-1}^{i} f(i) \mathrm{d}x \\ &\le \mathbf{1}_{j \ge 2} \sum_{i=1}^{j-1} \int_i^{i+1} f(x) \mathrm{d}x + f(y) + f(y) + \sum_{i=j+2}^{\infty} \int_{i-1}^{i} f(x) \mathrm{d}x \\ &= \mathbf{1}_{j \ge 2} \int_1^{j} f(x) \mathrm{d}x + 2f(y) + \int_{j+1}^{\infty} f(x) \mathrm{d}x \\ &< 2f(y) + \int_1^{\infty} f(x) \mathrm{d}x \le \frac{2e^{-\beta}}{\beta} + \int_1^{\infty} f(x) \mathrm{d}x, \end{aligned} \tag{46}$$

where the first inequality follows since f is increasing on $(1,j)$ and decreasing on $(j+1,\infty)$. We now estimate the integral. With a change of variables $u=\log x$, $\mathrm{d}v=e^{-\beta x}\mathrm{d}x$, we obtain

$$\int_1^\infty f(x)\mathrm{d}x=\left(-\frac{1}{\beta}\log x e^{-\beta x}\right)\Big|_1^\infty+\frac{1}{\beta}\int_1^\infty \frac{1}{x}e^{-\beta x}dx=\frac{1}{\beta}\mathtt{E}_1(\beta)<\frac{e^{-\beta}}{\beta}\log\left(1+\frac{1}{\beta}\right).$$

Combining with (46) and subsituting $\beta=-\log p$, it follows that

$$\begin{aligned}\sum_{i=1}^{\infty} f(i) &< \frac{p}{-\log p}\left[2+\log\left(1+\frac{1}{-\log p}\right)\right]\\ &< \frac{p}{1-p}\left[2+\log\left(1+\frac{1}{1-p}\right)\right]\\ &< \frac{p}{1-p}\left[2+\log\frac{2}{1-p}\right]\\ &< \frac{p}{1-p}\left[3+\log\frac{1}{1-p}\right], \qquad (47)\end{aligned}$$

where the second inequality is because $-\log p>1-p$ for $0<p<1$. Substituting (47) into (45) yields that

$$\begin{aligned}\Phi &\le p(1-p)+p^2(1-p)\log_2 3+\frac{2p^2}{\log 2}\left(3+\log\frac{1}{1-p}\right)\\ &\le p(1-p)+2p^2+\frac{6p^2}{\log 2}+\frac{2p^2}{\log 2}\log\frac{1}{1-p}\\ &= p(1-p)+\left(\frac{6}{\log 2}+2\right)p^2+\frac{2}{\log 2}p^2\log\frac{1}{1-p}.\end{aligned}$$

This concludes the proof. ∎

Appendix C - Proof of Lemma 3

Proof: $\Gamma(n,a)$ can be expanded as $\Gamma(n,a)=(n-1)!e^{-a}\sum_{k=0}^{n-1}\frac{a^k}{k!}$ [11]. When $0<a\le 1$, $\Gamma(n,a)\le(n-1)!e^{-a}\sum_{k=0}^{n-1}\frac{1}{k!}\le n!e^{-a}$; when $\alpha>1$, $\Gamma(n,a)\le(n-1)!e^{-a}\sum_{k=0}^{n-1}\alpha^k\le(n-1)!e^{-a}\sum_{k=0}^{n-1}\alpha^{n-1}=n!e^{-a}\alpha^{n-1}$. Therefore, $\Gamma(n,a)\le\max\{n!e^{-a},n!e^{-a}\alpha^{n-1}\}\le n!e^{-a}+n!e^{-a}\alpha^{n-1}=n!e^{-a}\left(1+\alpha^{n-1}\right)$. ∎

Appendix D - Proof of Lemma 4

Proof: The left hand side of (44) can be rewritten as $\left|\mathbf{u}^\dagger\mathbf{G}\mathbf{u}\right|$, where $\mathbf{G}=\mathbf{v}\mathbf{v}^\dagger-\mathbf{w}\mathbf{w}^\dagger$. Therefore, it is upper-bounded by the maximum value of $\left||\mathbf{u}^\dagger\mathbf{v}|^2-|\mathbf{u}^\dagger\mathbf{w}|^2\right|$ with respect to $\mathbf{u}$, which is the maximum absolute value for the singular value of $\mathbf{G}$. Using Gram-Schmidt orthogonalization, we obtain $\mathbf{v}_{\perp} = \frac{\mathbf{w}-\mathbf{v}\mathbf{v}^{\dagger}\mathbf{w}}{\sqrt{1-|\mathbf{v}^{\dagger}\mathbf{w}|^2}}$, which satisfies $|\mathbf{v}_{\perp}^{\dagger}\mathbf{v}_{\perp}|^2 = 1$ and $\mathbf{v}^{\dagger}\mathbf{v}_{\perp} = 0$. Then, $\mathbf{w}$ can be rewritten as $\mathbf{w} = \mathbf{v}\mathbf{v}^{\dagger}\mathbf{w} + \sqrt{1-|\mathbf{v}^{\dagger}\mathbf{w}|^2}\mathbf{v}_{\perp}$. Therefore, $\mathbf{G} = \left(1-|\mathbf{v}^{\dagger}\mathbf{w}|^2\right)\mathbf{v}\mathbf{v}^{\dagger} + \left(|\mathbf{v}^{\dagger}\mathbf{w}|^2-1\right)\mathbf{v}_{\perp}\mathbf{v}_{\perp}^{\dagger}$ and $\mathbf{G}\mathbf{G}^{\dagger} = \left(1-|\mathbf{v}^{\dagger}\mathbf{w}|^2\right)\mathbf{v}\mathbf{v}^{\dagger} + \left(1-|\mathbf{v}^{\dagger}\mathbf{w}|^2\right)\mathbf{v}_{\perp}\mathbf{v}_{\perp}^{\dagger}$. Since $1-|\mathbf{v}^{\dagger}\mathbf{w}|^2 \geq 0$, the maximum absolute singular value of $\mathbf{G}$ is $\sqrt{1-|\mathbf{v}^{\dagger}\mathbf{w}|^2}$. ■

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