COMBINATORIAL THEORY **3 (3)** (2023), #12 combinatorial-theory.org

A SYMMETRIC FUNCTION LIFT OF TORUS LINK HOMOLOGY

Andy Wilson[1]

[1]*Department of Mathematics, Kennesaw State University, Marietta, GA, U.S.A.*
awils342@kennesaw.edu

Submitted: Jun 29, 2022; Accepted: Oct 26, 2023; Published: Dec 22, 2023

Abstract. Suppose M and N are positive integers and let $k = \gcd(M, N)$, $m = M/k$, and $n = N/k$. We define a symmetric function $L_{M,N}$ as a weighted sum over certain tuples of lattice paths. We show that $L_{M,N}$ satisfies a generalization of Hogancamp and Mellit's recursion for the triply-graded Khovanov–Rozansky homology of the M, N-torus link. As a corollary, we obtain the triply-graded Khovanov–Rozansky homology of the M, N-torus link as a specialization of $L_{M,N}$. We conjecture that $L_{M,N}$ is equal (up to a constant) to the elliptic Hall algebra operator $\mathbf{Q}_{m,n}$ composed k times and applied to 1.

Keywords. Lattice paths, Dyck paths, link homology, torus links, elliptic Hall algebra, LLT polynomials

Mathematics Subject Classifications. 05E05, 57M27

1. Introduction

For coprime positive integers m and n, much has been discovered in recent years about the relationship between m, n-torus knots, the elliptic Hall algebra, and m, n-Dyck paths (lattice paths from $(0, 0)$ to (m, n) staying above the line $my = nx$). More precisely, the relevant objects are

(1) the triply-graded Khovanov–Rozansky homology of the m, n-torus knot,

(2) a certain symmetric function operator $\mathbf{Q}_{m,n}$ (defined in Section 2.2) applied to 1, and

(3) a generating function over m, n-Dyck paths, weighted by variables q and t as well as monomials in variables $x_1, x_2, x_3 \ldots$.

Gorsky and Negut conjectured that (2) and (3) are equal up to a sign [GN15]. This conjecture was proved by Mellit [Mel21]. An earlier result of Gorsky implies that (1) appears as a certain specialization of (2) and (3) [Gor12].

Somewhat less explored, but still fairly well understood, is the case where $m = n$, i.e. m and n are "minimally coprime." In this case, the objects are

(I) the triply-graded Khovanov–Rozansky homology of the n, n-torus link (no longer a knot),

(II) the symmetric function ∇p_{1^n}, where ∇ is F. Bergeron's Macdonald eigenoperator, and

(III) a generating function over an arbitrary number of labeled boxes placed into n columns [Wil18].

(II) and (III) are conjectured to be equal [Wil18]. The same specialization as before allows one to move from (II) or (III) to (I), making use of a recursion of Elias and Hogancamp [EH18].

The goal of this work is to generalize both the $\gcd(m, n) = 1$ and $m = n$ cases to any positive integers M and N. We let $k = \gcd(M, N)$, $m = M/k$, and $n = N/k$. The objects we consider are now

(A) the triply-graded Khovanov–Rozansky homology of the M, N-torus link,

(B) the elliptic Hall algebra operator $\mathbf{Q}_{m,n}$ applied iteratively k times to 1, and

(C) a generating function over k-tuples of (variations of) m, n-Dyck paths.

The correct analog of Elias and Hogancamp's recursion for (A) is known due to Hogancamp and Mellit [HM19].

The structure of this paper is as follows. In Section 2, we give the necessary background on link homology and the elliptic Hall algebra to understand objects (A) and (B) above. We develop the combinatorics of (C) in Section 3. In Section 4, we prove our main result, which is a recursion for our combinatorial generating functions. We also show how to specialize (C) to recover (A). We close by stating our main conjecture in Section 5, a relationship between (B) and (C), and explaining how the conjecture extends previous work.

2. Background

We suppose throughout the sequel that M and N are positive integers and let $k = \gcd(M, N)$, $m = M/k$, and $n = N/k$. In this section, we define the relevant notions from link homology theory and symmetric functions.

2.1. Torus link homology

A *link* is a topological subspace of $\mathbb{R}^3$ whose connected components are homeomorphic to circles. A connected link is called a *knot*. The goal of link invariants is to assign a mathematical object to each link (such as a number, a polynomial, or the homology of some chain complex) such that if two links are equivalent (usually by ambient isotopy) then the links are assigned the same object. In the other direction, if two links are assigned the same object, the links may or may not equivalent. One of the main goals of the theory is to derive link invariants for which this reverse implication holds as often as possible.

We are particularly interested in Khovanov–Rozansky homology, which assigns a triply-graded homology to each link [Kho07b, DBN02]. Khovanov–Rozansky homology generalizes many well-known link invariants such as the Alexander and Jones polynomials [Kho07a]. Instead of computing Khovanov–Rozansky homology precisely, we will focus on computing the related trivariate generating function using variables q, t, and a.

We will focus on *torus links*, a particular class of links. One way to depict a torus link is to begin by depicting the torus as $[0,1] \times [0,1]$ with vertical and horizontal edges identified, respectively. Given integers M and N with $\gcd(M,N) = k$, the M,N-torus link has components $My = Nx + \epsilon i$ for $i = 0$ to $k-1$ and some small value ϵ, where each component wraps around the torus until it forms a knot.

Hogancamp and Mellit derive a recursion for computing the triply graded Khovanov–Rozansky homology of any M,N-torus link [HM19]. We describe their recursion below, following Gorsky, Mazin, and Vazirani's variation [GMV20]. In their recursion and throughout this paper, v and w will be sequences in the alphabet $\{0, 1, \bullet\}$ with $|v| = |w|$, where $|v|$ indicates the the number of 1's in v.

Theorem 2.1 ([GMV20, HM19])**.** *For nonnegative integers M and N, the triply graded Khovanov–Rozansky homology of the M,N-torus link is free over $\mathbb{Z}$ of graded rank $p(0^M, 0^N)$, which is an element of $\mathbb{N}[q, t^{\pm 1}, a, (1-q)^{-1}]$ computed by the following recursion:*

0. $p(\bullet^M, \bullet^N) = 1$.

1. $p(\bullet v, \bullet w) = p(v\bullet, w\bullet)$.

2. $p(0v, 0w) = (1-q)^{-1} p(v1, w1)$ *if* $|v| = |w| = 0$.

3. $p(0v, 0w) = t^{-\ell} p(v1, w1) + q t^{-\ell} p(v0, w0)$ *if* $\ell = |v| = |w| > 0$.

4. $p(1v, 0w) = p(v1, w\bullet)$.

5. $p(0v, 1w) = p(v\bullet, w1)$.

6. $p(1v, 1w) = (t^{|v|} + a) p(v\bullet, w\bullet)$.

Hogancamp and Mellit's original recursion can be obtained by removing all $\bullet$'s from v and w. Our main result (Theorem 4.1) is a lift of Theorem 2.1 to the level of symmetric functions.

2.2. Symmetric functions

We let Λ denote the ring of symmetric functions in variables $x_1, x_2, x_3, \dots$ over the ground field $\mathbb{Q}(q,t,a,z)$. We use traditional notation for the usual bases for Λ such as the power sum, elementary, monomial, Schur, and Macdonald symmetric functions [Mac95]. We will often employ *plethystic substitution*, i.e. for any formal sum of Laurent monomials $A = a_1 + a_2 + a_3 + \dots$ from the x_i's or the ground field, we define

$$p_k[A] = a_1^k + a_2^k + a_3^k + \dots$$

for any power sum polynomial p_k. We extend plethystic substitution to all of Λ by viewing the p_i's as algebraically independent generators for Λ [LR11]. We also let

$$X = x_1 + x_2 + x_3 + \dots.$$

Our first definition is an important operator on Λ in the study of Macdonald polynomials.

Definition 2.2. For any nonnegative integer k and any symmetric function f,

$$\mathbf{D}_k f[X] = f[X + (1-q)(1-t)/z] \sum_{i \geqslant 0} (-z)^i e_i[X] \Big|_{z^k}$$

where $|_{z^k}$ extracts the coefficient of z^k from the series to its left.

Next, we define the fundamental operators for the elliptic Hall algebra.

Definition 2.3. For nonnegative integers m and n and $f \in \Lambda$, we define $\mathbf{Q}_{m,n} f$ by first setting

$$\mathbf{Q}_{0,n} f = \frac{qt}{qt-1} h_n[(1-qt)X/(qt)] \cdot f$$
$$\mathbf{Q}_{1,n} f = \mathbf{D}_n f.$$

Otherwise, $m \geqslant 2$. We assume m and n are coprime, so there are unique integers $1 \leqslant a < m$ and $1 \leqslant b < n$ such that $na - mb = 1$. Then we let

$$\mathbf{Q}_{m,n} f = \frac{1}{(1-q)(1-t)} [\mathbf{Q}_{m-a,n-b}, \mathbf{Q}_{a,b}] f.$$

$\mathbf{Q}_{m,n}$ is also defined when m and n are not coprime, but we will not need this level of generality [BGLX15]. $\mathbf{Q}_{m,n}$ applied to 1 appears in the Rational Shuffle Theorem [Mel21]. Our next goal is to develop a conjectured combinatorial formula for $\mathbf{Q}^k_{m,n}(1)$. In order to state our formula, we will use the following operators defined by Carlsson and Mellit in their proof of the Shuffle Theorem.

Definition 2.4. For any integer $\ell \geqslant 0$, we let

$$V_\ell = \mathbb{Q}(q,t)[y_1, y_2, \ldots, y_\ell] \otimes \Lambda.$$

Following Carlsson and Mellit [CM18], we define operators

$$d_+ : V_\ell \to V_{\ell+1} \qquad (\ell \geqslant 0)$$
$$d_- : V_\ell \to V_{\ell-1} \qquad (\ell \geqslant 1)$$

by

$$d_+ f = T_1 T_2 \ldots T_\ell f[X + (t-1) y_{\ell+1}]$$
$$d_- f = -y_\ell f[X - (t-1) y_\ell] \sum_{i \geqslant 0} h_i[-X/y_\ell] \Big|_{y_\ell^0}$$

where

$$T_i f = \frac{(t-1) y_i f + (y_{i+1} - t y_i) s_i f}{y_{i+1} - y_i}$$

and s_i swaps y_i and y_{i+1}. We define a third operator,

$$d_= : V_\ell \to V_\ell \qquad (\ell \geqslant 1)$$

which also appears in Carlsson and Mellit's work but not by this name. It acts by

$$d_= f = \frac{1}{t-1} \left(d_- d_+ f - d_+ d_- f \right).$$

Now we are ready to describe the relevant combinatorial objects.

3. Combinatorial objects

Again, for positive integers M and N we let $k = \gcd(M, N)$, $m = M/k$, and $n = N/k$. We will define a generating function as a sum over certain k-tuples of lattice paths. These lattice paths will depend on a pair of sequences v and w of lengths M and N, respectively, in the alphabet $\{0, 1, \bullet\}$. Furthermore, we let $|v|$ be the number of 1's in v and insist that $|v| = |w|$. We will denote the resulting generating function by $L(v, w)$.

3.1. Path tuples

Definition 3.1. An m, n-*path*, or just a *path*, is a sequence of n unit-length north and m unit-length east steps from $(a, 0)$ to $(a + m, n)$ for some integer $a \leqslant 0$ that

- begins with a north step, and
- stays weakly above the line $my = nx$.

Any path (as just defined) can be transformed into an infinite path in the plane by copying and pasting the path at shifts of $(\ell m, \ell n)$ for every integer ℓ. Any height-n "band" (region between $y = b$ and $y = b + n$ for some integer b) of the infinite path determines the original path. We will most often work with the $b = 0$ band. Next, we define a labeling of the unit squares in each of k "sheets" of $(\mathbb{Z}^2)^k$.

Definition 3.2 ([And02])**.** We consider $(\mathbb{Z}^2)^k$ as k *sheets* of $\mathbb{Z}^2$, where the sheets are indexed from 0 to $k - 1$. The *content* of a lattice square (or "cell") in the i^{th} sheet, where $0 \leqslant i < k$, is

$$i + My - Nx$$

where (x, y) are the coordinates of the lower right lattice point of the square.

We depict the contents of some cells in Figure 3.1. Content provides a bijective correspondence between the cells in any band of $(\mathbb{Z}^2)^k$ and $\mathbb{Z}$. As a result, given a fixed band, we can refer to a cell by its content. In the sequel, we often use the phrase "cell c" to refer to the unique cell in the current band with content equal to c.

A:

B:

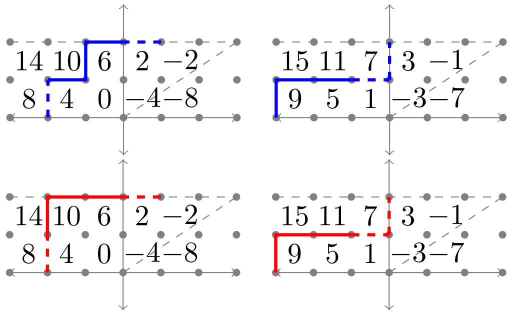

Figure 3.1: We have drawn the only two possible path tuples for $v = 000110$ and $w = 0110$, labeled A (in blue) and B (in red). We have $M = 6$ (the length of v), $N = 4$ (the length of w), and $k = \gcd(6, 4) = 2$, which is why each tuple has 2 sheets. The dotted lines denote basement steps. Path tuple A has area 0 and 4 path diagonal inversions, corresponding to the (c, d) pairs $(3, 6), (4, 6), (4, 9), (6, 9)$. Path tuple B has area 1 (from cell 10) and also has 4 path diagonal inversions, corresponding to pairs $(3, 6), (4, 6), (4, 9), (9, 10)$. The finite complements (in $\mathbb{Z}_{\geqslant 0}$) of the invariant sets corresponding to A and B are $\{0, 1, 2, 3, 4, 5, 6, 9\}$ and $\{0, 1, 2, 3, 4, 5, 6, 9, 10\}$, respectively.

Definition 3.3. For a fixed k-tuple of paths $\boldsymbol{P}$ in sheets $0, 1, \ldots, k-1$ of $\mathbb{Z}^2$, every cell $c \in \mathbb{Z}$ in a fixed band of $\boldsymbol{P}$ is either

- *above* $\boldsymbol{P}$, i.e. the north step of $\boldsymbol{P}$ in c's row is to the right of c and the east step of $\boldsymbol{P}$ in c's column is below c, or
- *below* $\boldsymbol{P}$.

Of the cells c below $\boldsymbol{P}$, a cell c

- has a *north step in* $\boldsymbol{P}$ if there is a north step of $\boldsymbol{P}$ on c's left boundary and
- has an *east step in* $\boldsymbol{P}$ if an east step of $\boldsymbol{P}$ is on c's upper boundary.

If c is below $\boldsymbol{P}$ but it does not have a north step nor an east step in $\boldsymbol{P}$, then c is *strictly below* $\boldsymbol{P}$.

Next, we use sequences v and w in the alphabet $\{0, 1, \bullet\}$ of lengths M and N, respectively, such that $|v| = |w|$, to restrict the set of paths we consider. These sequences v and w govern the relative location of the paths and the cells $0, 1, \ldots, M-1$ and $0, 1, \ldots, N-1$, respectively.

Definition 3.4. A *v, w-path tuple* (or just a *path tuple*) is a k-tuple of m, n-paths $\boldsymbol{P} = (P^{(0)}, P^{(1)}, \ldots, P^{(k-1)})$, one in each sheet, such that, for $v = v_0 \ldots v_{M-1}$,

(i) $v_c = \bullet$ if and only if cell c is above $\boldsymbol{P}$,

(ii) $v_c = 1$ if and only if cell c is a north step of $\boldsymbol{P}$, and

(iii) $v_c = 0$ if and only if neither (i) nor (ii) hold.

and, for $w = w_0 \ldots w_{N-1}$,

(i') $w_c = \bullet$ if and only if cell c is above $\boldsymbol{P}$,

(ii') $w_c = 1$ if and only if cell c is an east step of $\boldsymbol{P}$, and

(iii') $w_c = 0$ if and only if neither (i') nor (ii') hold.

North steps in cells $0 \leqslant c < M$ and east steps in cells $0 \leqslant c < N$ are called *basement steps* and denoted with dashed lines in our figures.

The previous definition can also be used to recover v and w from a given path tuple by checking where each cell c for $0 \leqslant c < M$ and $0 \leqslant c < N$ appears in relation to the tuple. As a sanity check, we note that, for any M and N, the restriction that paths are weakly above the line $my = nx$ implies that there is exactly one path tuple if $v = \bullet^M$ and $w = \bullet^N$. On the other hand, if $v = 0^M$ and $w = 0^N$, there are infinitely many v, w-path tuples. Figure 3.1 contains two examples of path tuples.

Although we have insisted that $|v| = |w|$ without much justification, this condition is actually necessary in order for v, w-path tuples to exist. In other words, if $|v| \neq |w|$ then there are no tuples of paths that meet the conditions in Definition 3.4[1]. Although we will not use this fact in our results, it is worthwhile to consider why this should be true. Note that any north basement step in a cell $0 \leqslant c < M$ must be immediately preceded by an east step (in the infinite version of the path). This east step occurs in a cell with content $c - (M - N) < N$, so this is an east basement step. Similarly, any east basement step must be followed immediately by a north basement step. This argument provides an outline for a bijection between north basement steps and east basement steps in any path.

3.2. Invariant sets

There is a class of objects known as M, N-*invariant sets*, or just *invariant sets*, that is in bijection with path tuples. These objects appear in work of Gorsky, Mazin, and Vazirani [GMV20].

Definition 3.5. An M, N-*invariant set* is a set $\Delta \subset \mathbb{Z}_{\geqslant 0}$ with finite complement in $\mathbb{Z}_{\geqslant 0}$ such that, for every $i \in \Delta$, $i + M \in \Delta$ and $i + N \in \Delta$. We let $I_{M,N}$ denote the collection of all M, N-invariant sets.

Given any k-tuple of m, n-paths $\boldsymbol{P}$, there is a natural partner invariant set Δ given by

$$i \in \Delta \iff i \text{ is above } \boldsymbol{P}.$$

In fact, this is a bijection, since $\boldsymbol{P}$ can be recovered from Δ. If we wish to consider v, w-path tuples $\boldsymbol{P}$, we get information about the intersections $\Delta \cap \{0, 1, \ldots, M-1\}$ and $\Delta \cap \{0, 1, \ldots, N-1\}$.

Definition 3.6. An invariant set Δ *fits* sequences v and w if its corresponding k-tuple of m, n-paths $\boldsymbol{P}$ is a v, w-path.

We will occasionally use this identification with invariant sets in our proofs, although we will primarily work with path tuples. Next, we define statistics for path tuples.

[1]This observation is thanks to an anonymous referee.

3.3. Path tuple statistics

Definition 3.7. The *area* of a path tuple $\boldsymbol{P}$ is the number of cells $c \geqslant M + N$ that are below $\boldsymbol{P}$.

Next, we define a notion of diagonal inversion for a path.

Definition 3.8. A *path diagonal inversion* in a path tuple $\boldsymbol{P}$ is a pair of cells $c < d$ with $c \geqslant 0$ and $M \leqslant d < c + M$ such that

- c has a north step in $\boldsymbol{P}$ and
- d is (weakly) below $\boldsymbol{P}$.

c and d may be in different sheets and the north step mentioned above can be a basement step. We let $\operatorname{pdinv}(\boldsymbol{P})$ denote the number of path diagonal inversions in a path tuple $\boldsymbol{P}$.

3.4. The characteristic function of a path tuple

Given a path tuple $\boldsymbol{P}$, we describe how to obtain a sequence of d_+, d_-, and $d_=$ operators that, when applied to 1, allow us to define $L(v, w)$. We also note that, in this subsection, we will break the M, N-symmetry that has existed thus far.

Definition 3.9. A *partial Schröder path* is a lattice path from $(0, \ell)$ to (r, r) for some nonnegative integers $\ell \leqslant r$ consisting of steps $(+1, 0)$, $(0, +1)$ and $(+1, +1)$ that remains weakly above the line $y = x$ and does not contain any diagonal steps on the line $y = x$.

Definition 3.10. Given a path tuple $\boldsymbol{P}$, suppose $0 \leqslant c_1 < c_2 < \ldots < c_r$ are the nonnegative cells containing north steps in $\boldsymbol{P}$ and that c_ℓ contains the last basement north step, i.e. $c_\ell < M \leqslant c_{\ell+1}$. (If there are no basement north steps, let $\ell = 0$.) Place the values $c_1, c_2, \ldots, c_r$ in the first r squares on the line $y = x$ from bottom left to top right. Let S be the (unique) partial Schröder path from $(0, \ell)$ to (r, r) such that

(i) if $c_i < c_j < c_i + M$, then the entire (unique) square above c_i and to the left of c_j is below S,

(ii) if $c_i + M = c_j$, then the square above c_i and to the left of c_j contains a diagonal step,

(iii) otherwise, the square above c_i and to the left of c_j is above S.

We call S the *partial Schröder path* of $\boldsymbol{P}$.

Definition 3.11. Given the partial Schröder path S of a path tuple $\boldsymbol{P}$, we begin with 1 and, reading S from right to left, iteratively apply

- d_+ for any horizontal step,
- d_- for any vertical step, and
- $d_=$ for any diagonal step.

The resulting element of V_ℓ is the *characteristic function* of $\boldsymbol{P}$, written $\chi(\boldsymbol{P})$.

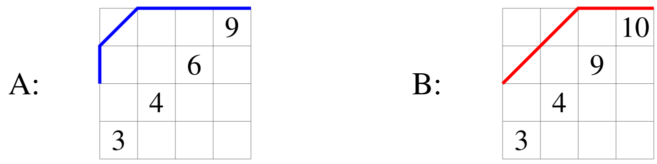

Figure 3.2: This figure contains the partial Schröder paths of the two path tuples from Figure 3.1. Their corresponding characteristic functions are $d_-d_=d_+d_+d_+(1)$ and $d_=d_=d_+d_+(1)$, both of which are in V_2.

The partial Schröder paths and characteristic functions for the path tuples in Figure 3.1 are depicted in Figure 3.2. We close this subsection by defining our main objects of study.

Definition 3.12. Given sequences v and w of lengths M and N, respectively, and $|v| = |w|$, we let

$$L(v,w) = \sum_{\boldsymbol{P}} t^{-\operatorname{pdinv}(\boldsymbol{P})} q^{\operatorname{area}(\boldsymbol{P})} \chi(\boldsymbol{P})$$

where the sum is over all v, w-path tuples $\boldsymbol{P}$.

Example 3.13. To complete this subsection, we compute $L(v,w)$ in its entirety for $v = 000110$ and $w = 0110$. There are two path tuples in this case, which appear in Figure 3.1. As mentioned in Figure 3.1, the areas of the path tuples are 0 and 1, respectively, and both tuples have 4 path diagonal inversions. Figure 3.2 depicts the partial Schröder paths for these path tuples. Assembling this information, and evaluating the corresponding sequences of Carlsson–Mellit operators, we get

$$\begin{aligned} L(000110, 0110) &= t^{-4} d_- d_= d_+ d_+ d_+(1) + t^{-4} q d_= d_= d_+ d_+(1) \\ &= -t^{-2} y_1 s_1 + t^{-3} q y_1 y_2 \in V_2 \end{aligned}$$

where s_1 is the Schur function of degree 1.

3.5. Path labelings when $v = 0^M$ and $w = 0^N$

We can simplify the previous definitions for $L(v,w)$ when $v = 0^M$ and $w = 0^N$. As a result, we get an expression for $L(v,w)$ that does not explicitly use Carlsson–Mellit operators. We can assign labels to north steps in our paths in a manner reminiscent of parking functions.

Definition 3.14. A *labeled path tuple* is a k-tuple of paths $\boldsymbol{P}$ along with a function f from the north steps of $\boldsymbol{P}$ to the positive integers such that, if $c + M = d$, then $f(c) < f(d)$.

We define a notion of diagonal inversions for labeled path tuples.

Definition 3.15. Given a labeled path tuple $(\boldsymbol{P}, f)$, a *labeled diagonal inversion* is a pair of cells c and d with north steps such that $c < d < c + M$ and $f(c) < f(d)$. (Again, c and d may be in different sheets.) We let $\operatorname{ldinv}(\boldsymbol{P}, f)$ be the number of labeled diagonal inversions in the labeled path tuple $(\boldsymbol{P}, f)$.

Definition 3.16. We let

$$L_{M,N} = \sum_{(\boldsymbol{P},f)} t^{\mathrm{ldinv}(\boldsymbol{P},f)-\mathrm{pdinv}(\boldsymbol{P})} q^{\mathrm{area}(\boldsymbol{P})} \prod_{i>0} x_i^{|f^{-1}(i)|}$$

where the sum is over all labeled path tuples $(\boldsymbol{P}, f)$.

It is reasonable to ask how the power of t here relates to the codinv statistic defined by Gorsky, Mellit, and Vazirani in the context of Theorem 2.1 [GMV20]. To relate to the two approaches directly, each labeling f should be strictly increasing up the line $y = x$. Suppose u is a square below S, above the square corresponding to cell c, and to the right of the square corresponding to cell d (i.e. c and d are nonnegative north steps in the path tuple). Suppose also that $v = 0^M$ and $w = 0^N$. Then the following hold:

1. If (c, d) is a labeled diagonal inversion then it is also a path diagonal inversion.
2. (c, d) is a path diagonal inversion but not a labeled diagonal inversion if and only if $(c + N, d + N)$ is counted by the codinv statistic for invariant sets [GMV20, Definition 2.5].

This connects our statistics to those of Gorsky, Mazin, and Vazirani up to a global shift by the value they denote $\delta(N, M)$ [GMV20, (2)]. Outside of the case $v = 0^M$ and $w = 0^N$, we cannot make any exact statements about the relationship between our statistics and theirs.

To conclude this section, we make note of a theorem which comes directly from Definiton 3.16 and the work of Carlsson and Mellit [CM18]. In fact, results like these are what motivated the original definition of Carlsson and Mellit's d_+ and d_- operators.

Theorem 3.17. *For positive integers M and N,*

$$L_{M,N} = L(0^M, 0^N).$$

While it is theoretically possible to unwind the definitions of the Carlsson–Mellit operators to give a label-based expression for $L(v, w)$ for any v and w, the resulting expression is quite technical and not helpful in what we aim to achieve here. Understanding this expression may be a worthwhile endeavor in the future.

4. A recursion for $L(v, w)$

Now we are able to prove our main result, a recursion for $L(v, w)$, which contains the rank generating function of the Khovanov–Rozansky torus link homology of any torus link as a specialization.

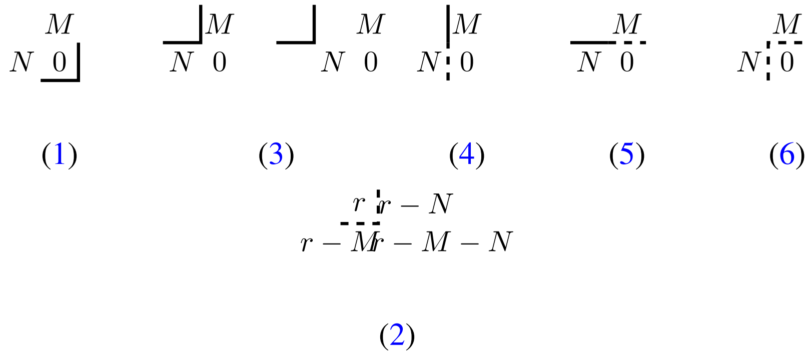

Figure 4.1: We sketch the different possible configurations of steps near cells 0, M, and N in a path tuple $\boldsymbol{P}$ which correspond to the different recursive cases in Theorem 4.1.

Theorem 4.1. *Let v and w be sequences in the alphabet $\{0, 1, \bullet\}$. Assume in each case that both indexing sequences have the same number of 1's. We can compute $L(v, w)$ via the following recursion:*

0. $L(\bullet^M, \bullet^N) = 1.$

1. $L(\bullet v, \bullet w) = L(v\bullet, w\bullet).$

2. $L(0v, 0w) = (1-q)^{-1} d_- L(v1, w1)$ *if* $|v| = |w| = 0.$

3. $L(0v, 0w) = t^{-\ell} d_- L(v1, w1) + qt^{-\ell} L(v0, w0)$ *if* $\ell = |v| = |w| > 0.$

4. $L(1v, 0w) = t^{-\ell} d_= L(v1, w\bullet)$ *if* $\ell = |v| = |w| - 1.$

5. $L(0v, 1w) = L(v\bullet, w1).$

6. $L(1v, 1w) = d_+ L(v\bullet, w\bullet)$

Proof. Throughout the proof, we will let $\boldsymbol{P}$ be a path tuple contributing to the left-hand side of one of the equations in the recursion. (0) holds since there is exactly one such $\boldsymbol{P}$ – the path tuple corresponding to the invariant set $\mathbb{Z}_{\geqslant 0}$ – and it does not have any area or path diagonal inversions. Furthermore, since all of its north steps occur in cells < 0, its characteristic function is trivially 1. Therefore (0) is proven.

For the remaining parts, we will describe bijections between path tuples $\boldsymbol{P}$ that contribute to the left-hand side of each equation and path tuples $\boldsymbol{P}'$ that contribute to the right-hand side of each part. Throughout, let Δ be the invariant set corresponding to $\boldsymbol{P}$. In general, the invariant set Δ' of $\boldsymbol{P}'$ will be formed by decrementing some of the entries of the invariant set Δ corresponding to $\boldsymbol{P}$, although this formulation will vary slightly depending on the part. $\boldsymbol{P}$ is a path tuple for sequences iv and jw for some symbols i and j. i and j describe where the cell 0 appears in relation to the north and east steps of $\boldsymbol{P}$, respectively. Figure 4.1 is included to help the reader visualize the ways $\boldsymbol{P}$ interacts with cells 0, M, and N in each case.

To prove (1), we note that cell 0 is above $\boldsymbol{P}$, so cells M and N must also be above $\boldsymbol{P}$. In other words, $0, M, N \in \Delta$. Let $\boldsymbol{P}'$ be the path tuple corresponding to the invariant set $\Delta' = \{a-1 : a \in \Delta \setminus \{0\}\}$. Then $M-1$ and $N-1$ are above $\boldsymbol{P}'$, so $\boldsymbol{P}'$ is a $v\bullet, w\bullet$-path tuple. Furthermore, the area, inversions, and characteristic function of $\boldsymbol{P}'$ are equal to that of $\boldsymbol{P}$, so the contribution of $\boldsymbol{P}'$ to $L(v\bullet, w\bullet)$ matches the contribution of $\boldsymbol{P}$ to $L(\bullet v, \bullet w)$.

For all remaining statements except (2), which we will prove last, $\boldsymbol{P}'$ is the path tuple whose invariant set Δ' is $\{a-1 : a \in \Delta\}$. Since the case where $0 \in \Delta$ was already dealt with in (0) and (1), we have $0 \notin \Delta$ and $\Delta' \subset \mathbb{Z}_{\geqslant 0}$.

We skip (2) for now and prove (3). In (3), 0 is strictly below $\boldsymbol{P}$ but there is some cell $1 \leqslant c < M-1$ that is a north step of $\boldsymbol{P}$. The "corner" (east step followed by a north step) on the horizontal line on 0's north edge either appears at the upper left vertex of the 0 cell or to the left of this vertex. If it appears at the upper left vertex, then M contains a north step and N contains an east step. These steps become basement steps at $M-1$ and $N-1$, respectively, in $\boldsymbol{P}'$. In particular, $M-1$ is a basement north step in $\boldsymbol{P}'$ while M is a non-basement north step in $\boldsymbol{P}$. This explains the appearance of d_- in the first summand on the right-hand side of (3). If, on the other hand, the corner above 0 does not occur at the upper left vertex of cell 0, $i = j = 0$ and we do not see a new d_- in this characteristic function. Furthermore, cell $M+N$ counts toward the area of $\boldsymbol{P}$ but not that of $\boldsymbol{P}'$, leading to the q factor in the rightmost term of (3). Finally, in either case, there are ℓ path diagonal inversions of the form (c, M) in $\boldsymbol{P}$, one for each basement north step in $\boldsymbol{P}$, that are not path diagonal inversions in $\boldsymbol{P}'$, since the second coordinate in the pair is no longer at least M. This is why $t^{-\ell}$ appears next to both terms on the right in (3).

In (4), 0 has a north step but not an east step. Therefore M has a north step and N is above $\boldsymbol{P}$. In $\boldsymbol{P}'$, $M-1$ has a north step and $N-1$ is above the path. Furthermore, there are ℓ path diagonal inversions of the form (c, M) in $\boldsymbol{P}$ that are not path diagonal inversions in $\boldsymbol{P}'$. Finally, the partial Schröder path of $\boldsymbol{P}$ has a diagonal step formed by 0 and M that does not exist in the partial Schröder path of $\boldsymbol{P}'$, since $0-1=-1$ does not affect the partial Schröder path of $\boldsymbol{P}'$.

In (5), 0 has an east step but not a north step. Similarly, $M-1$ above $\boldsymbol{P}'$ and $N-1$ has an east step in $\boldsymbol{P}'$. Since M is not below $\boldsymbol{P}$, there are no path diagonal inversions of the form (c, M), this statistic is unchanged by decrementing, as is the area and characteristic function.

In (6), 0 has a north step and an east step, which means that, in $\boldsymbol{P}'$, $M-1$ and $N-1$ are both above the path. 0 appears at the bottom of the partial Schröder path for $\boldsymbol{P}$ but -1 cannot occur in the partial Schröder path of $\boldsymbol{P}'$, which accounts for the d_+ in (6).

Finally, we return to (2). We know that none of the cells $0, 1, \ldots, M-1$ are north steps in $\boldsymbol{P}$ and none of the cells $0, 1, \ldots, N-1$ are east steps in $\boldsymbol{P}$. It must be the case, then, that all these cells are strictly below $\boldsymbol{P}$ and that every cell $c < M+N$ must be below $\boldsymbol{P}$. In other words, we must have $v = 0^{M-1}$ and $w = 0^{N-1}$. Let $r \geqslant M+N$ be the smallest cell above $\boldsymbol{P}$, so $r-N$ and $r-M$ are north and east steps of $\boldsymbol{P}$, respectively. Let $\boldsymbol{P}'$ be the path tuple corresponding to $\Delta' = \{a-(r-M-N+1) : a \in \Delta\}$. (Note that this is a different construction from prior cases.) $\boldsymbol{P}'$ has a north step in cell $(r-N)-(r-M-N+1) = M-1$ and an east step in cell $(r-M)-(r-M-N+1) = N-1$. Suppose $\boldsymbol{P}'$ had a north step in some cell $c < M-1$. Then $c+(r-M-N+1)$ is less than $(M-1)+(r-M-N+1) = r-N$, which has a north step in $\boldsymbol{P}$, so $c+(r-M-N+1)+N < r$ is above $\boldsymbol{P}$, which contradicts the choice of r. Similarly, suppose $\boldsymbol{P}'$ had an east step in some cell $d < N-1$. Since $d+(r-M-N+1)$ is less than $(N-1)+(r-M-N+1) = r-M$, which has an east step in $\boldsymbol{P}$, $d+(r-M-n+1)+M < r$ is above $\boldsymbol{P}$, which is another contradiction. Therefore $\boldsymbol{P}'$ does contribute to $L(0^{M-1}1, 0^{N-1}1)$, as desired.

We still have to check that the $(1-q)^{-1}d_-$ factor is correct in (2). First, $M-1$ is has basement north step in $\boldsymbol{P}'$, whereas its corresponding cell $(M-1)+(r-M-N+1) = r-N$ contains a non-basement north step in $\boldsymbol{P}$. Therefore a leading d_- indeed differentiates their respective characteristic functions. Finally, we want to show that decrementing every cell $r-M-N+1$ times lowers the area by $r-M-N$. Since $r \geqslant M+N$, every time we decrement, we know that $M+N$ is above the path, since if it is not we would have chosen a smaller r. After we decrement, the cell $M+N-1$ no longer contributes to the area statistic, since that statistic only counts cells $\geqslant M+N$. This argument holds until the last time we decrement, since in this case $M+N$ is already above the path before decrementing. Therefore we lose $r-M-N$ area while moving from $\boldsymbol{P}$ to $\boldsymbol{P}'$. Since any $r \geqslant M+N$ is possible, we can lose any nonnegative integer amount of area in this process. This accounts for the appearance of the power series term $(1-q)^{-1}$ in (2). □

To illustrate the proof of Theorem 4.1, let us consider the case $L(000110, 0110)$ for which both possible path tuples are depicted back in Figure 3.1. Recall that the partial Schröder paths of these tuples appear in Figure 3.2. The first entries of both sequences are 0 but the sequences are not entirely zero, so we are in case (3) of Theorem 4.1. Let A' and B' be the result of decrementing (the invariant sets corresponding to) path tuples A and B, respectively. Note that cell $M=6$ has a non-basement north step for path tuple A but, after decrementing, cell 5 has a basement north step in A'. This is why a d_- appears in the first term on the right of (3). On the other hand, path tuple B contributes to the second term on the right. Note that 10 is an area cell for path tuple B but $10-1=9$ is not an area cell for B', yielding the q factor in the second term. Finally, note that we have lost $\ell = 2$ path diagonal inversions in each case while moving from A to A' and from B to B'. Decrementing the path diagonal inversions $(4,9)$ and $(6,9)$ in A yields path diagonal inversions $(3,8)$ and $(5,8)$ in A', while decrementing the path diagonal inversions $(3,6)$ and $(4,6)$ yields pairs $(2,5)$ and $(3,5)$, which cannot be path diagonal inversions in A' because $5 < M = 6$. A similar phenomenon holds for B and B'.

One can view Theorem 4.1 as a lift of Theorem 2.1 to the level of symmetric functions (or, more precisely, symmetric functions tensored with polynomials in variables $y_1, y_2, \ldots, y_\ell$). The result below shows how to recover any $p(v,w)$ from the corresponding $L(v,w)$. In path tuple A, the corner above cell 0 is at the upper left vertex of cell 0.

It is important to note that it is only at the level of $p(0^M, 0^N)$ that these objects are link invariants – in particular, $L_{M,N}$ is *not* a link invariant. One easy way to see this is to note that the $1,N$-torus link is equivalent to the unknot for any N, while $L_{1,N}$ is a symmetric function of degree N, and thus must vary with N. While reading an earlier draft of this paper, one of the anonymous referees suggested $L_{M,N}$ is an invariant of the M,N-torus link considered inside a filled torus instead of inside $\mathbb{R}^3$. We believe it would be valuable to pursue this idea in future work.

Corollary 4.2. *Let ψ be the operator*[2] *on Λ defined by $\psi(e_i) = 1 + a$ for each (algebraically independent) e_i. Then, for any sequences v and w in the alphabet $\{0, 1, \bullet\}$ with $|v| = |w| = \ell$,*

$$\psi\left(d_-^{\ell} L(v, w)\right) = p(v, w).$$

Proof. We work by induction in the order induced by the recursions in Theorems 2.1 and 4.1. That is, given v and w, we find the case of Theorem 4.1 for which v and w index the expression on the left of the equation. We assume the corollary is already known for the expression on the right of the same equation. Since this order is indeed a total order on pairs of such sequences, as proven by Gorsky, Mazin, and Vazirani using "decision trees" [GMV20], and the corollary certainly holds for the base case in which $v = \bullet^M$ and $w = \bullet^N$, this will prove the corollary.

The recursive steps (0) and (1) are directly equal. From (2), we have

$$\begin{aligned}\psi\left(L(0^M, 0^N)\right) &= \psi\left((1-q)^{-1} d_- L(0^{M-1}1, 0^{N-1}1)\right)\\ &= (1-q)^{-1}\psi\left(d_- L(0^{M-1}1, 0^{N-1}1)\right)\\ &= (1-q)^{-1} p(0^{M-1}1, 0^{N-1}1)\\ &= p(0^M, 0^N)\end{aligned}$$

by induction and Theorems 2.1 and 4.1. If $|v| = |w| = \ell > 0$, we compute

$$\begin{aligned}\psi\left(d_-^{\ell} L(0v, 0w)\right) &= \psi\left(t^{-\ell} d_-^{\ell+1} L(v1, w1) + q t^{-\ell} d_-^{\ell} L(v0, w0)\right)\\ &= t^{-\ell}\psi\left(d_-^{\ell+1} L(v1, w1)\right) + q t^{-\ell}\psi\left(d_-^{\ell} L(v0, w0)\right)\\ &= t^{-\ell} p(v1, w1) + q t^{-\ell} p(v0, w0)\\ &= p(0v, 0w).\end{aligned}$$

The case in (4) is more difficult due to the presence of the $d_=$ operator. We consider

$$\psi\left(d_-^{\ell+1} L(1v, 0w)\right) = \psi\left(t^{-\ell} d_-^{\ell+1} d_= L(v1, w\bullet)\right)$$

which we hope to show is equal to

$$p(1v, 0w) = p(v1, w\bullet) = \psi\left(d_-^{\ell+1} L(v1, w\bullet)\right).$$

We will approach this combinatorially using Haglund's work on shuffles and the zeta map, using the fact that

$$\psi(F) = \sum_{i=0}^{n} \langle F, h_i e_{n-i}\rangle a^i$$

for any symmetric function F of degree n [Hag08]. For any Schröder path S, it follows that we can write

$$\psi(\chi(S)) = \sum_f t^{\mathrm{inv}(f,S)} \prod f_i$$

where f is any labeling of the squares on the line $y = x$ below S with labels a and 1 such that, for a square u weakly below S, above f_i, and to the left of f_j,

[2] To experts in this area, applying ψ equivalent to taking the "Schröder inner product."

- if u contains a diagonal step of S then $f_i = 1$ and
- if u is strictly below S then f_i and f_j are not both a.

The statistic $\operatorname{inv}(f, S)$ counts squares u strictly below S that are above a 1 label.

Let S be any partial Schröder path that contributes to $L(v1, w\bullet)$. Then the full Schröder path $\mathtt{n}^{\ell+1}S$ (where $\mathtt{n}$ is a unit north step) contributes to $d_{\ell+1}L(v1, w\bullet)$ and $\mathtt{n}^{\ell+1}\mathtt{d}S$ contributes to $t^{-\ell}d_-^{\ell+1}d_=L(v1, w\bullet)$ (where $\mathtt{d}$ is a diagonal step). The labelings f for $\chi(\mathtt{n}^{\ell+1}S)$ are in bijection with those for $\chi(\mathtt{n}^{\ell+1}\mathtt{d}S)$, where the bijection is simply prepending a 1. This 1 forms an inversion with exactly the next ℓ entries of f, which cancels out the $t^{-\ell}$ factor in $t^{-\ell}d_-^{\ell+1}d_=L(v1, w\bullet)$.

The case in (5) is straightforward. Finally, we need to understand the appearance of d_+ in (6). In that case, we want to show that

$$\psi\left(d_-^{\ell+1}L(1v, 1w)\right) = \psi\left(d_-^{\ell+1}d_+L(v\bullet, w\bullet)\right)$$

is equal to

$$p(1v, 1w) = (t^\ell + a)p(v\bullet, w\bullet) = (t^\ell + a)\psi\left(d_-^\ell L(v\bullet, w\bullet)\right)$$

where $\ell = |v| = |w|$. To get a labeling f that contributes to the former equation, we simply prepend a 1 (yielding a factor of t^ℓ) or an a (yielding a factor of a) to get a labeling that contributes to the latter equation. □

5. A conjecture for $Q^k_{m,n}(1)$

In this section, we state our main conjecture and explain how it generalizes previously known results.

Conjecture 5.1. For positive integers M and N, let $k = \gcd(M, N)$, $m = M/k$, $n = N/k$. Then

$$\mathbf{Q}^k_{m,n}(1) = \pm(1-q)^k t^C L_{M,N}$$

where C is the maximum of $\operatorname{pdinv}(\boldsymbol{P})$ over all M, N-path tuples $\boldsymbol{P}$.

We currently do not know of a simple way to compute the value C in Conjecture 5.1, nor do we know a simple rule for the sign that appears.

5.1. The Rational Shuffle Theorem

When M and N are coprime (so $M = m$, $N = n$, and $k = 1$), the Rational Shuffle Theorem gives a combinatorial expression for $\mathbf{Q}_{m,n}(1)$. We check that our formulas agree. Since $k = 1$, there is only one sheet in each path tuple. We can decrement each path tuple until $M + N$ is the smallest value above the path, i.e. M is the lowest-content north step and N is the lowest-content east step. We obtain the usual depiction of rational parking functions by viewing such a path tuple in the horizontal band $1 < y \leqslant n + 1$.

5.2. The $M = N$ case

If $M = N = k$, then sheet i is a single row containing all the cells equivalent to i modulo k. We choose one cell as a north and east step in every sheet. This simplifies to the "unbounded columns" that appear in an open conjecture for $Q^k_{1,1}(1) = \nabla p_{1^k}$ [Wil18].

5.3. Conclusion

A reasonable approach to proving Conjecture 5.1 would be to mimic the recursion in Theorem 4.1 to define and study extensions of the operators $\mathbf{Q}_{m,n}$ to sequences v and w. A similar approach was used by Mellit to prove the Rational Shuffle Theorem [Mel21]. An alternative approach would be to try to take advantage of the powerful "Cauchy identity" for non-symmetric Hall–Littlewood polynomials, following Blasiak, Haiman, Morse, Pun, and Seelinger [BHM+23].

It would also be interesting to investigate whether $L(v, w)$ has a precise geometric or topological meaning outside of the $v = 0^M$ and $w = 0^N$ case. For example, Hogancamp and Mellit show that $p(1^k 0^{m(k-1)}, 1^k 0^{n(k-1)})$ is related to the colored homology of certain torus links [HM19].

We have assigned, to every M, N-torus link, a symmetric function $L_{M,N}$ with connections to Khovanov–Rozansky homology. It is not clear how one might extend this assignment to produce a symmetric function L_K for an arbitrary link K. Any progress in this direction would be quite exciting.

References

[And02] J. Anderson. Partitions which are simultaneously t_1- and t_2-core. *Discrete Math.*, 248:237–243, 2002. doi:10.1016/s0012-365x(01)00343-0.

[BGLX15] F. Bergeron, A. Garsia, E. S. Leven, and G. Xin. Compositional (km, kn)-Shuffle Conjectures. *Int. Math. Research Notices*, October 2015. doi:10.1093/imrn/rnv272.

[BHM+23] J. Blasiak, M. Haiman, J. Morse, A. Pun, and G. H. Seelinger. A shuffle theorem for paths under any line. *Forum of Mathematics, Pi*, 11, 2023. doi:10.1017/fmp.2023.4.

[CM18] E. Carlsson and A. Mellit. A proof of the shuffle conjecture. *J. Amer. Math. Soc.*, 31:661–697, 2018. arXiv:1508.06239, doi:10.1090/jams/893.

[DBN02] Dror D. Bar-Natan. On Khovanov's categorification of the Jones polynomial. *Alg. & Geom. Topology*, 2:337–370, 2002. doi:10.2140/agt.2002.2.337.

[EH18] B. Elias and Matthew Hogancamp. On the computation of torus link homology. *Comp. Math.*, 155:164–205, 2018. doi:10.1112/s0010437x18007571.

[GMV20] E. Gorsky, M. Mazin, and M. Vazirani. Recursions for rational q, t-Catalan numbers. *J. Combin. Theory Ser. A*, 173, 2020. doi:10.1016/j.jcta.2020.105237.

[GN15] E. Gorsky and A. Negut. Refined knot invariants and Hilbert schemes. *J. Math. Pures Appl.*, 9:403–435, 2015. doi:10.1016/j.matpur.2015.03.003.

[Gor12] E. Gorsky. q, t-Catalan numbers and knot homology. *Contemp. Math.*, 566:213–232, 2012. doi:10.1090/conm/566/11222.

[Hag08] J. Haglund. *The q, t-Catalan Numbers and the Space of Diagonal Harmonics*. Amer. Math. Soc., 2008. Vol. 41 of University Lecture Series.

[HM19] M. Hogancamp and A. Mellit. Torus link homology. September 2019. arXiv:1909.00418.

[Kho07a] M. Khovanov. Link homology and categorification. In *Proceedings of the International Congress of Mathematicians Madrid*, volume II, pages 989–999. EMS Press, 2007. arXiv:math/0605339, doi:10.4171/022-2/47.

[Kho07b] Mikhail Khovanov. Triply-graded link homology and Hochschild homology of Soergel bimodules. *Internat. J. Math.*, 18:869–885, 2007. doi:10.1142/s0129167x07004400.

[LR11] N. A. Loehr and J. B. Remmel. A computational and combinatorial exposé of plethystic calculus. *J. Alg. Comb.*, 33:163–198, 2011. doi:10.1007/s10801-010-0238-4.

[Mac95] I. Macdonald. *Symmetric Functions and Hall Polynomials*. Oxford University Press, second edition, 1995.

[Mel21] A. Mellit. Toric braids and (m, n)-parking functions. *Duke Math. J.*, 170:4123–4169, 2021. doi:10.1215/00127094-2021-0011.

[Wil18] A. T. Wilson. Torus link homology and the nabla operator. *J. Comb. Theory, Ser. A*, 154:129–144, February 2018. arXiv:1606.00764, doi:10.1016/j.jcta.2017.08.009.